Efficient Algorithms for Computing the Betti Numbers of Semi-algebraic Sets

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  - Dimension of the ambient space : k.

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- Recent work in complexity theory (Cucker, Buergisser) on the real version of counting complexity classes.
- Some ideas may be useful in designing algorithms for computing homology groups in other contexts.

Cylindrical Algebraic Decomposition

Effective method of decomposing semi-algebraic sets.

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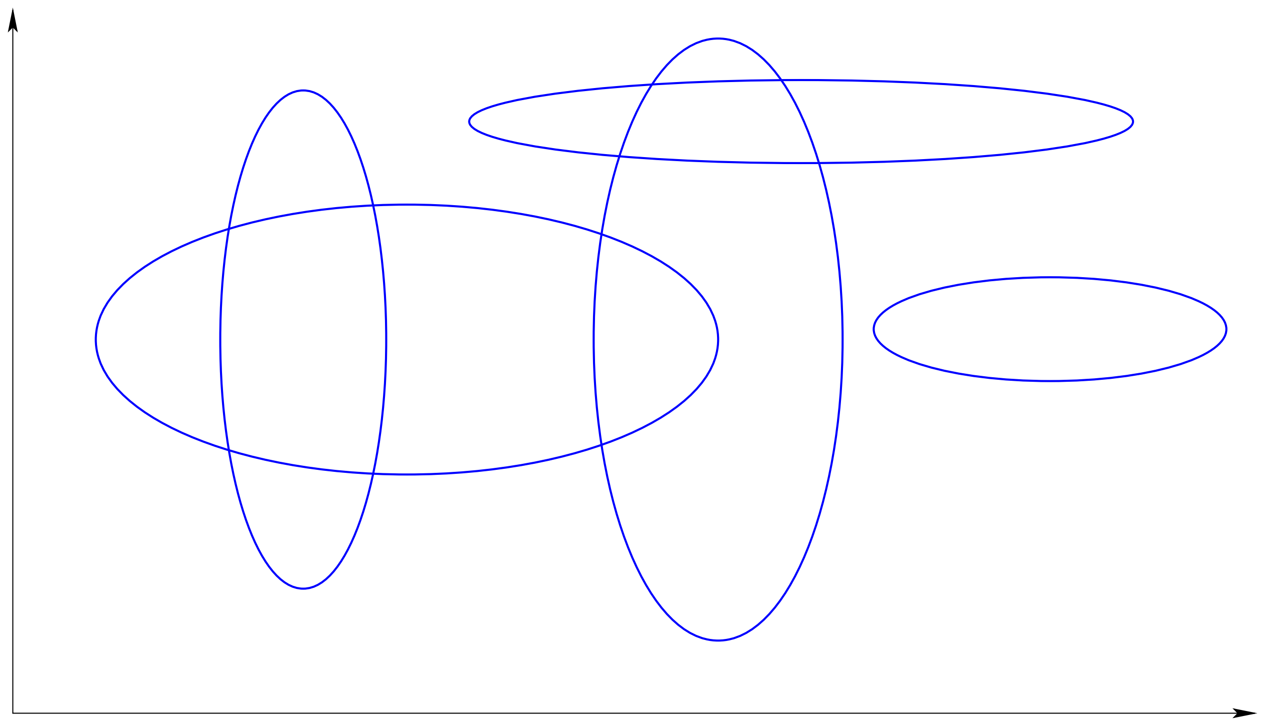

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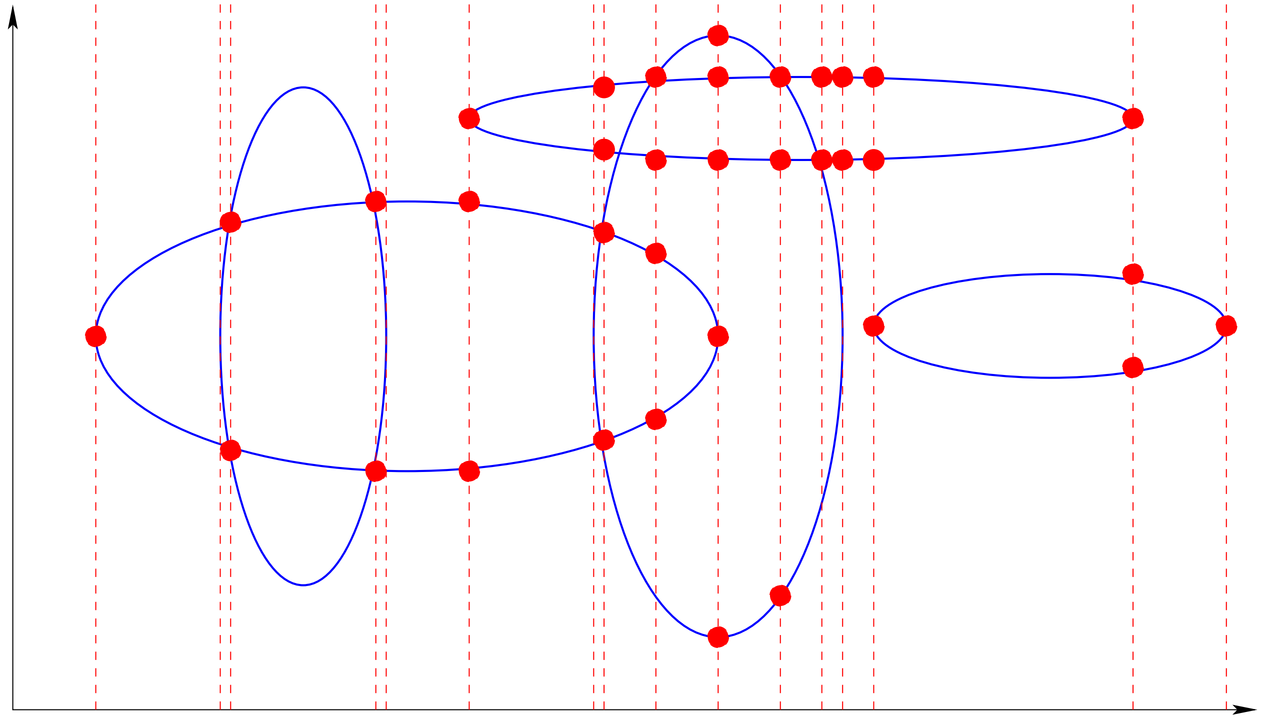

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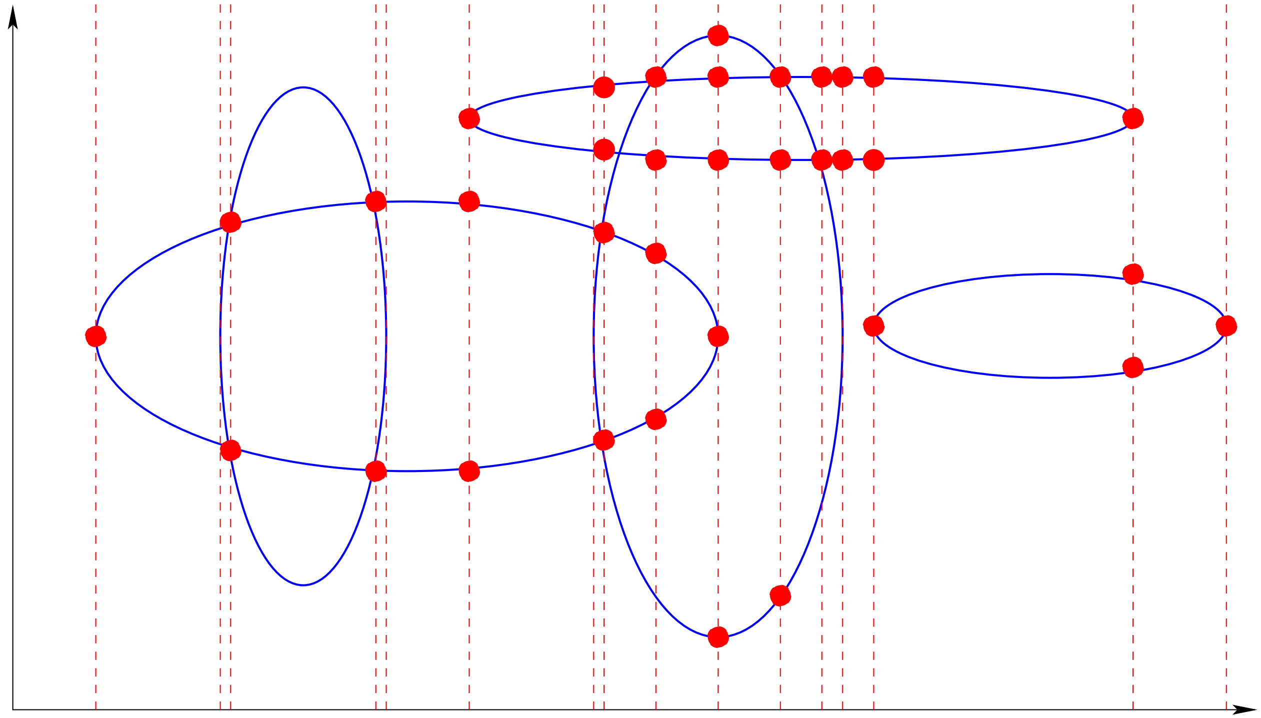

Complexity is double exponential in the dimension.

$$(O(nd))^{2^k}$$

Classical Result

Theorem 1 *(Oleinik and Petrovsky, Thom, Milnor) Let $S \subset R^k$ be the set defined by the conjunction of n inequalities,*

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Then,

$$\sum_i b_i(S) \ \leq \ nd(2nd-1)^{k-1} = O(nd)^k.$$

Tightness

The above bound is actually quite tight. Example: Let

$$P_i = L_{i,1}^2 \cdots L_{i,\lfloor d/2 \rfloor}^2 - \epsilon,$$

where the L_{ij}'s are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_n \geq 0$ has $\Omega(nd)^k$ connected components and hence $b_0(S) = \Omega(nd)^k$.

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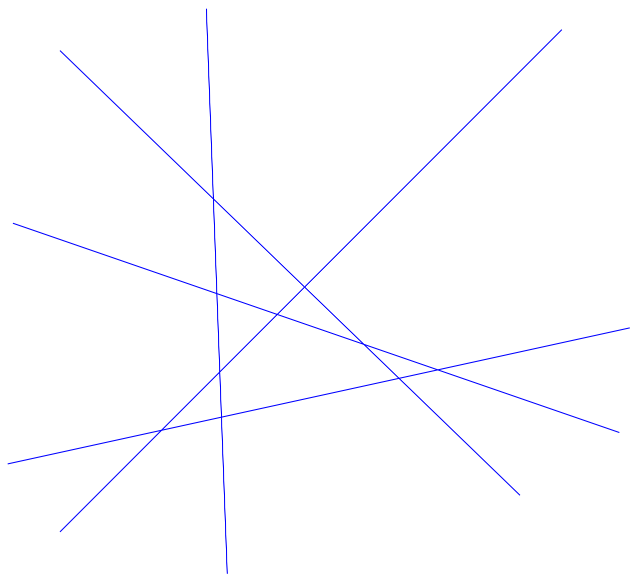

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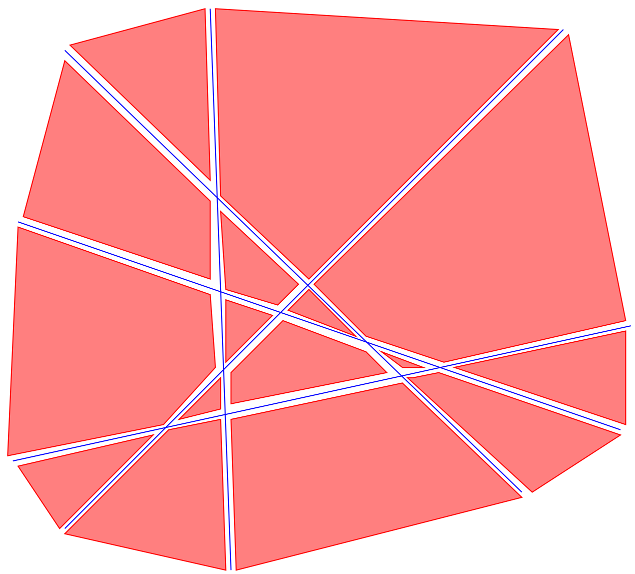

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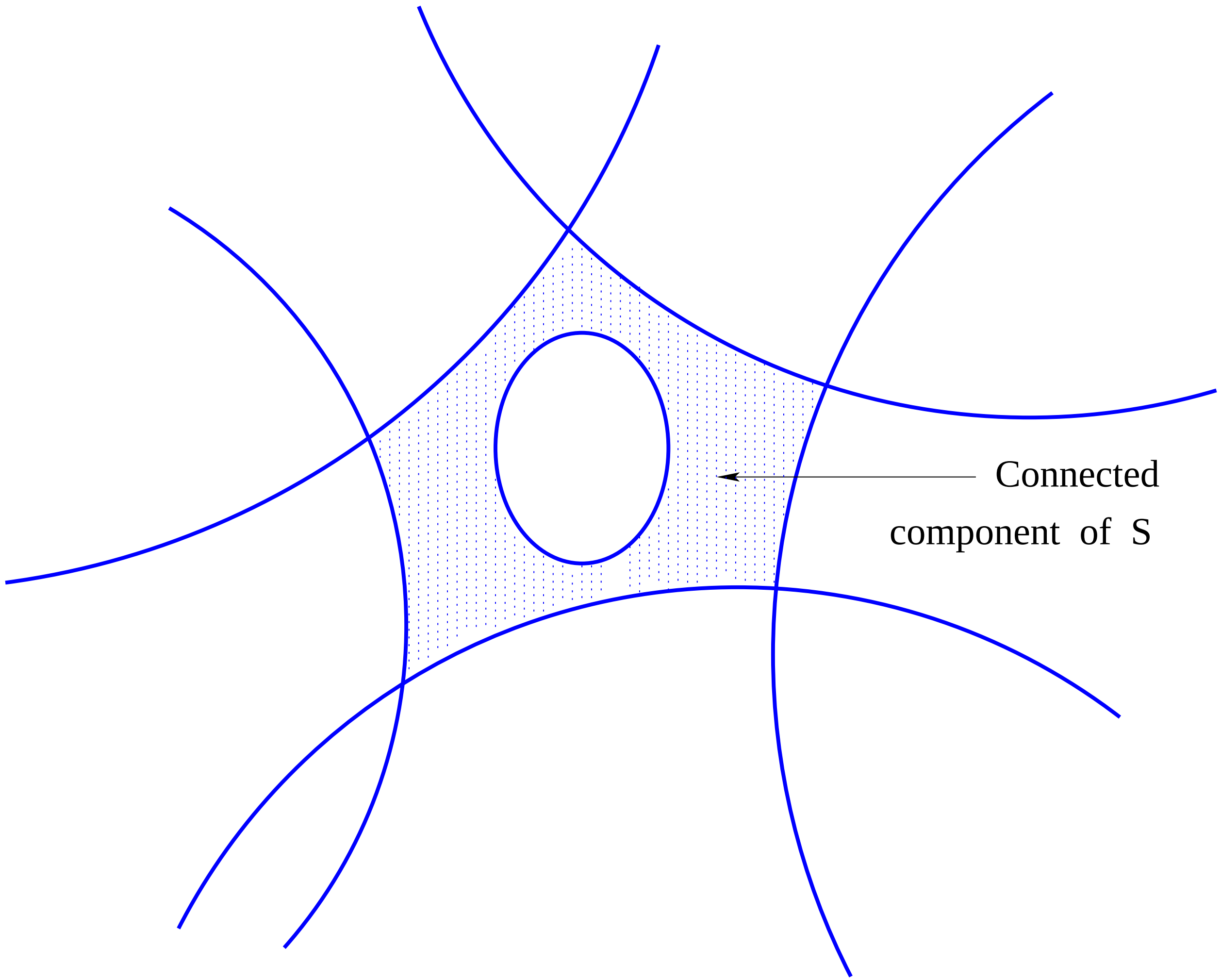
- Cannot construct examples such that $b_i(S) = \Omega(nd)^k$ for $i > 0$.
- The technique used for proving the above result does not help: Replace the semi-algebraic set S by another set bounded by a smooth algebraic hypersurface of degree nd having the same homotopy type as S. Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.

Picture Proof of Thom-Milnor Bound



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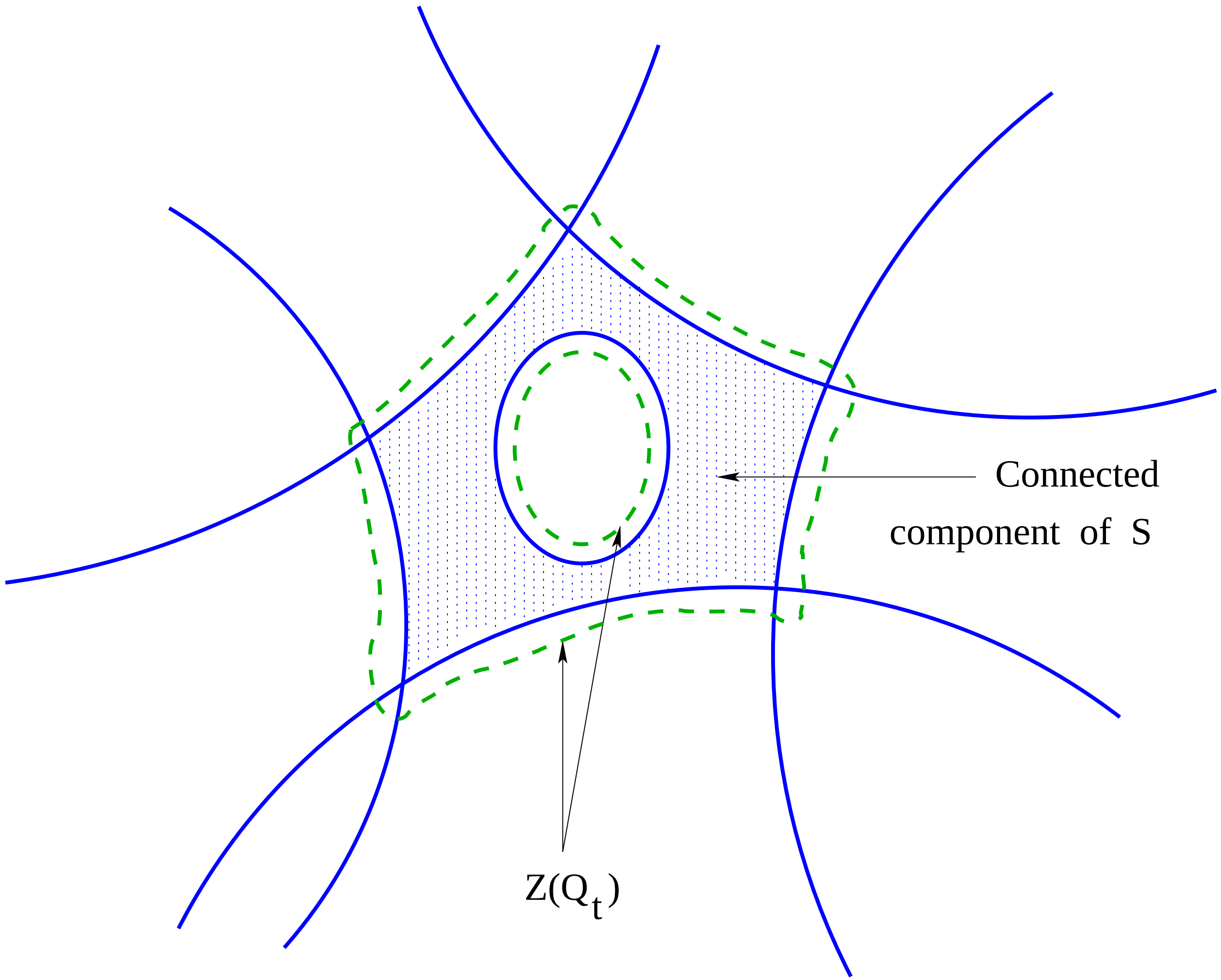

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- Double exponential vs single exponential vs polynomial time.
- Problems that can be solved in single exponential time: Testing emptiness, deciding connectivity, computing descriptions of the connected components, computing the Euler-Poincaré characteristic of a given semi-algebraic set.
- Problems for which no single exponential time algorithm is known: Computing the higher Betti numbers, computing semi-algebraic triangulations, or semi-algebraic stratifications.

New Results

- Single exponential time algorithm for computing the first Betti number of semi-algebraic sets (with R. Pollack, M-F. Roy).

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- Polynomial time algorithm to compute the top Betti numbers of semi-algebraic sets defined by quadratic inequalities.

Another approach

- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the -vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.

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- We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
- The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_{p+1}} C^*(A_{\alpha_0,\ldots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

Mayer-Vietoris Double Complex

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
0 & \longrightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0,\alpha_1,\alpha_2}) \\
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\end{array}
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\vdots & & \vdots & & \vdots & & \\
d \uparrow & & d \uparrow & & d \uparrow & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
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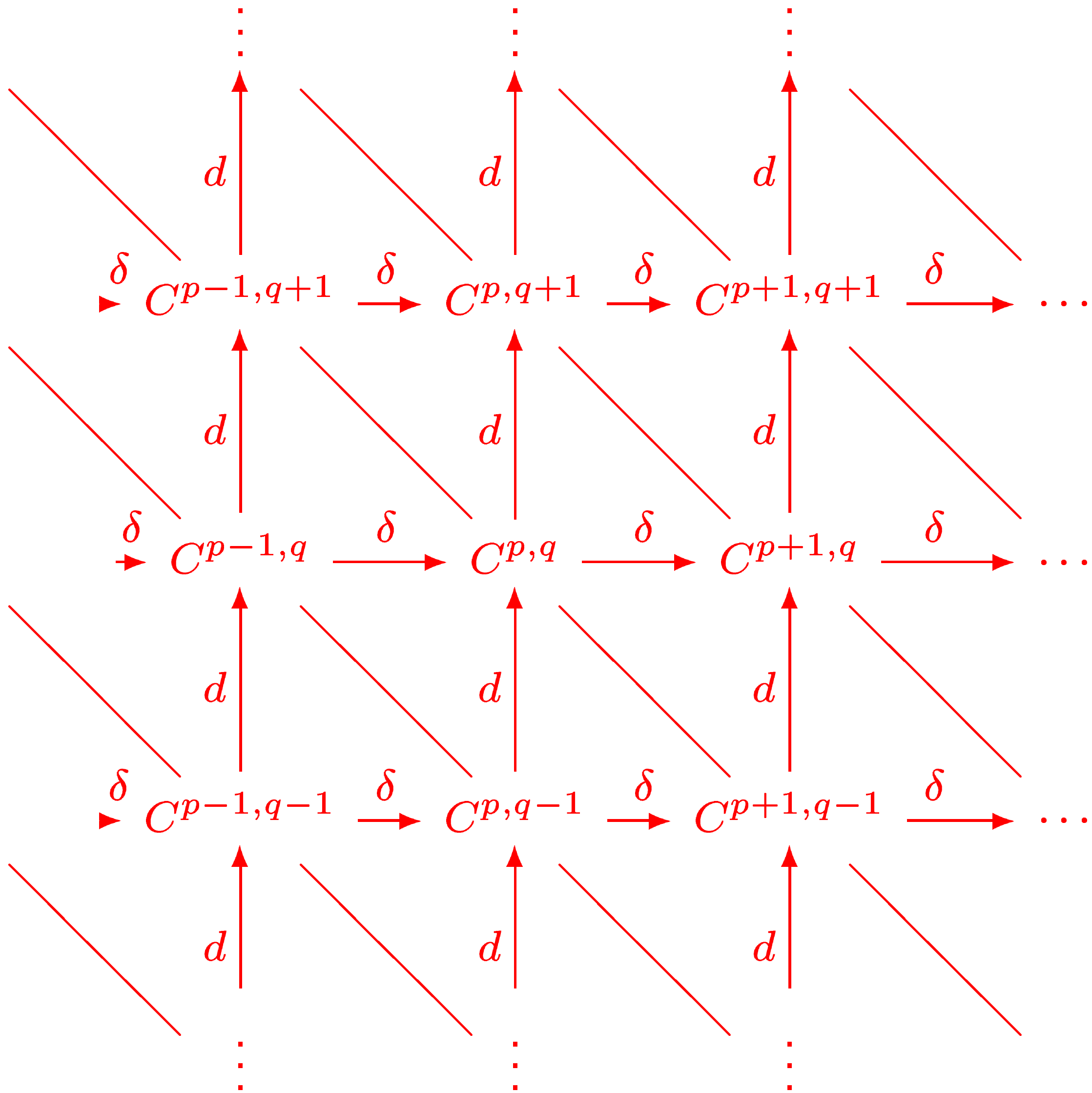
The Associated Total Complex



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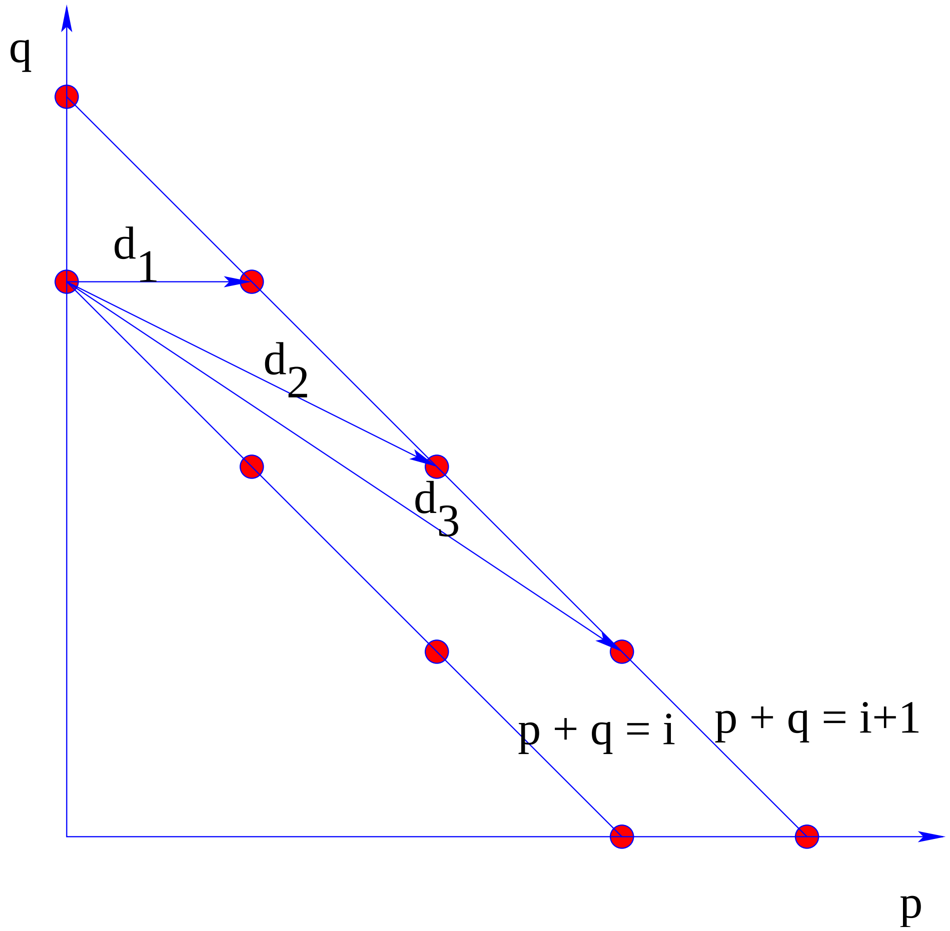
- $E_\infty = H^*(\text{Associated Total Complex})$.

Spectral Sequence



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- $$E_1' = H_d(\mathcal{M}), E_2' = H_\delta H_d(\mathcal{M})$$

E_1

$\vdots$	$\vdots$	$\vdots$
$C^3(A)$	0	0
$C^2(A)$	0	0
$C^1(A)$	0	0
$C^0(A)$	0	0

E_2

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
H^3(A) & 0 & 0 \\
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\end{array}$$

E_1'

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Inequality I

Let A be a finite simplicial complex and $A_1, \ldots, A_n$ sub-complexes of A such that $A = A_1 \cup \cdots \cup A_n$. Then for every $i \geq 0$,

$$b_i(A) \leq \sum_{j=1}^{i+1} \sum_{J \subset \{1,\ldots n\}, \#(J)=j} b_{i-j+1}(A_J),$$

where $A_J = \cap_{j \in J} A_j$.

Inequality II

Let be a real closd field and $V \subset^k$ be the set defined by the conjunction of ℓ inequalities,

$$P_1 \geq 0, \ldots, P_\ell \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$deg(P_i) \leq d, 1 \leq i \leq \ell,$$

contained in a variety $Z(Q)$ of real dimension k' with $deg(Q) \leq d$.
Then, for all $i,\ 0 \leq i \leq k'$,

$$b_i(V) \leq (3^\ell - 1)d(2d-1)^{k-1}.$$

Graded Bounds

Theorem 2 *(B, 2001) Let $S \subset R^k$ (resp. $T \subset R^k$) be the set defined by the conjunction (resp. disjunction) of n inequalities,*

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restricted to a variety $Z(Q)$ of real dimension k' with $\deg(Q) \leq d$. Then,

$$b_i(S) \leq \sum_{j=0}^{k'-i} \binom{n}{j} 2^{j+1} d(2d-1)^{k-1} = \binom{n}{k'-i} O(d)^k,$$

$$b_i(T) \leq \sum_{j=0}^{i+1} \binom{n}{j} 3^j d(2d-1)^{k-1} = \binom{n}{i+1} O(d)^k.$$

Sets defined by Quadratic Inequalities

Theorem 3 *(B, 2001) Let* ℓ *be any fixed number and let* $S \subset R^k$ *be defined by* $P_1 \geq 0, \ldots, P_n \geq 0$ *with* $\deg(P_i) \leq 2$*. Then,*

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This bound is *polynomial in the dimension k* unlike the O-P-T-M bound which was *single exponential in k*. Notice also that the lowest Betti numbers of S cannot be polynomially bounded. Example: S defined by

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Suppose that we can compute an acyclic covering in single exponential time. Then, all $H^i(A_j) = 0, i > 0$.
Then, $H_1(A)$ is isomorphic to the middle homology of,

$$\prod_{\alpha_0} H^0(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0,\alpha_1,\alpha_2})$$

Deciding Connectivity

Given two points x and y in a set S;

- Decide whether they are in the same connected component of S.
- If yes, construct a path in S joining them.
- (B-Pollack-Roy, 1995) We give an algorithm to solve both problems for semi-algebraic sets restricted to a variety of dimension k' in time,

$$n^{k'+1}d^{O(k^2)}.$$

- (B-Pollack-Roy, 1997) We also give semi-algebraic descriptions of the connected components in time

$$n^{k+1}d^{O(k^3)}.$$

What is a Roadmap ?

A roadmap of S, passing through a given set of points, $\mathcal{M}$, $R(S, \mathcal{M})$, is a semi-algebraic set of dimension at most one containing $\mathcal{M}$, satisfying:

1. for every semi-algebraically connected component C of S, $C \cap R(S, \mathcal{M})$ is non-empty and semi-algebraically connected.
2. for every $x \in R$, and for every semi-algebraically connected component C' of S_x, $C' \cap R(S, \mathcal{M})$ is non-empty, where $S_x = S \cap (X_1 = x)$.

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In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

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Step 1: Follow the X_2-extremal points in the X_1 direction.
Algebraically, follow parametrically the solutions of,

$$Q = \frac{\partial Q}{\partial X_3} = \cdots = \frac{\partial Q}{\partial X_k} = 0.$$

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In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

Step 2: Recurse at certain special slices corresponding to the critical values of the projection map onto the X_1 co-ordinate.

Algebraically, critical values are the X_1 co-ordinates of the real solutions of the system,

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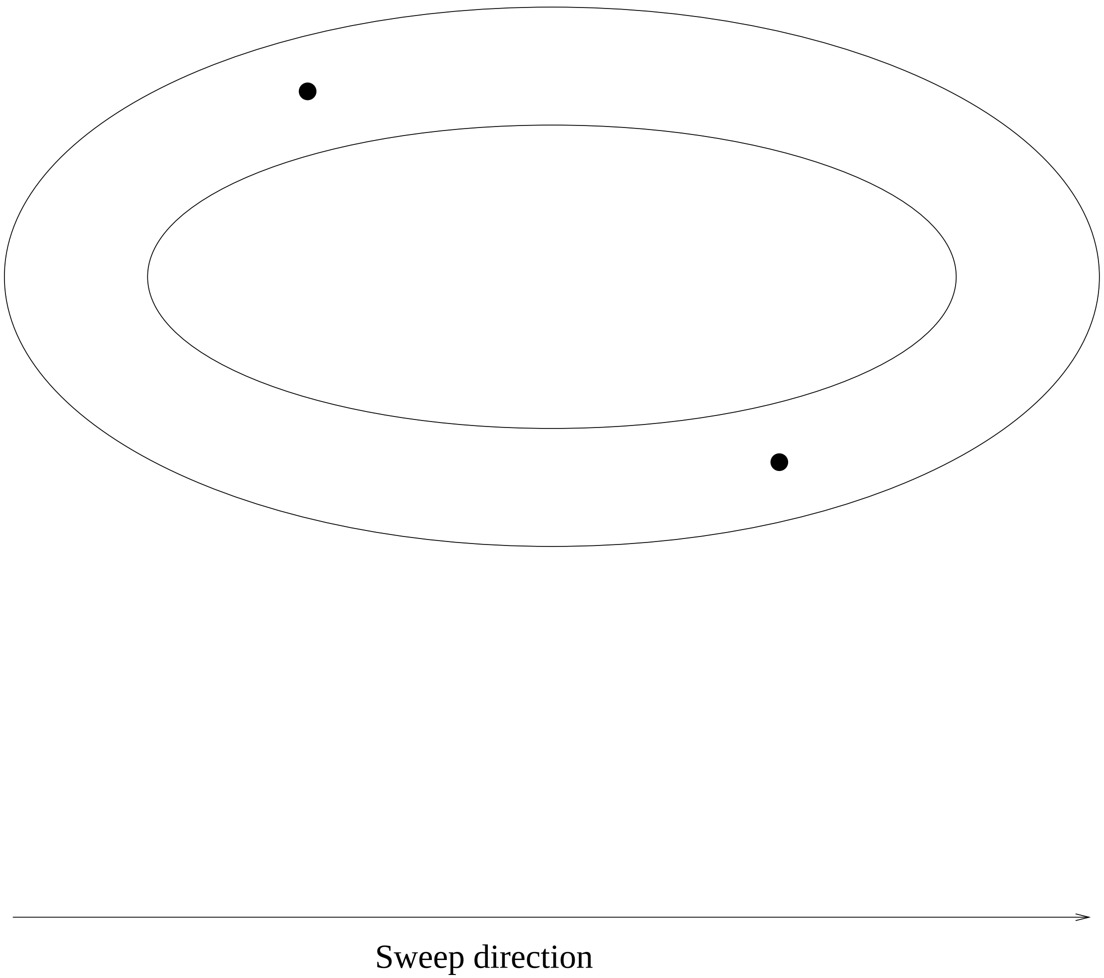
In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

Step 3: Recurse also at the X_1 co-ordinates of the input points.

The torus in R^3



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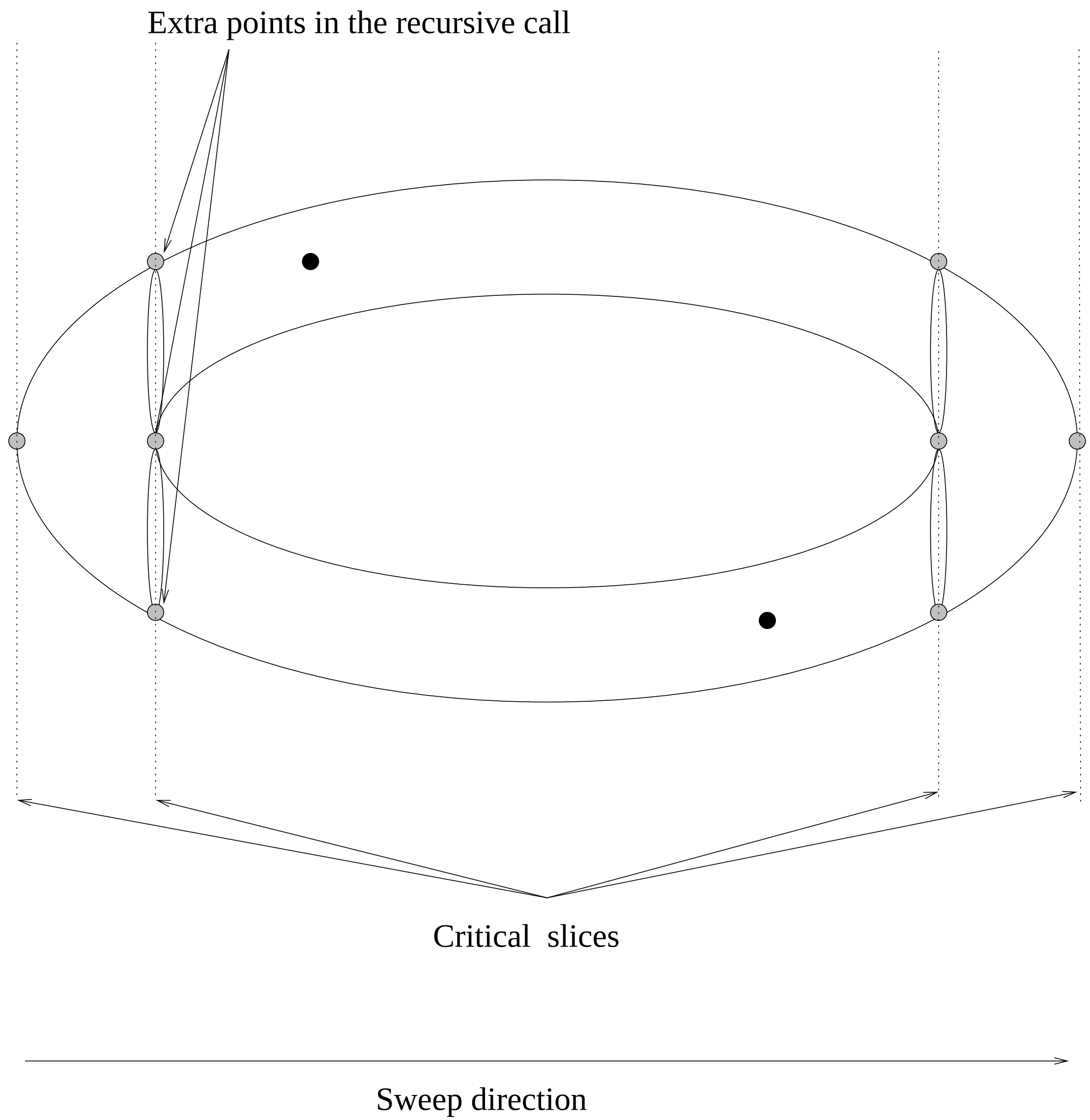

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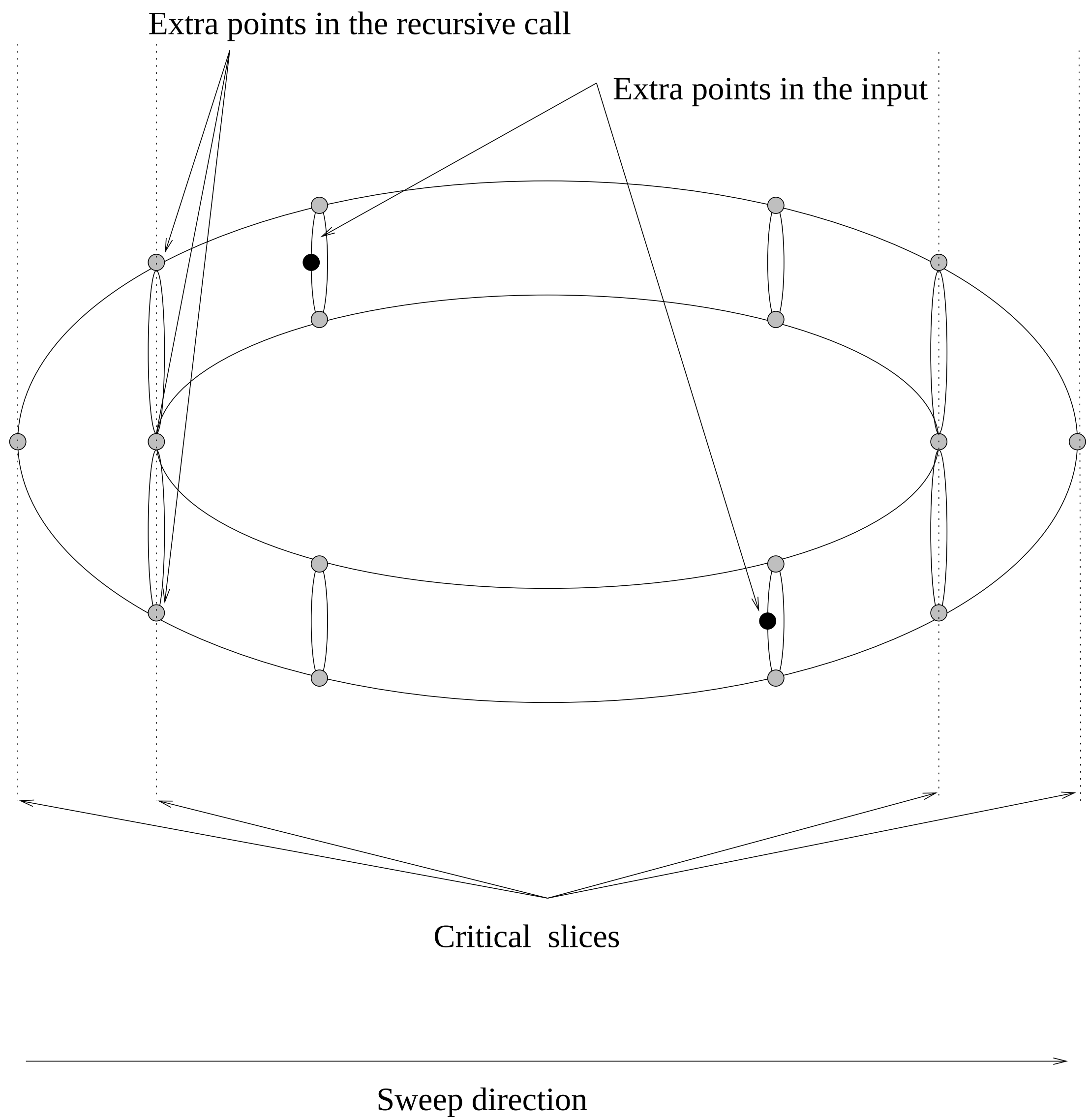

Connected Components and Contractible Cover

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- Points satisfying the same sign condition on $\mathcal{L}$ lie in the same connected component of S. Thus, each connected component of S can be described by a disjunction of sign conditions on $\mathcal{L}$.
- For a fixed sign condition σ on $\mathcal{L}$, the union of the paths $\Gamma(y)$ such that $\mathrm{sign}\mathcal{L}(y) = \sigma$ is contractible.

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Algorithm for sets defined by quadratic inequalities

For any fixed $\ell > 0$, there is an algorithm which given a set of n polynomials,

$$\mathcal{P} = \{P_1, \ldots, P_n\} \subset [X_1, \ldots, X_k],$$

with

$$\deg(P_i) \leq 2, 1 \leq i \leq n,$$

computes

$$b_k(S), \ldots, b_{k-\ell}(S),$$

where S is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is

$$s^{\ell+2} k^{2^{O(\ell)}}.$$

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- Consider S as the intersection of the various S_i's and consider the double complex arising from the generalized Mayer-Vietoris exact sequence.
- This enables us to reduce the problem of computing the top ℓ Betti numbers of S, to the problem of computing certain complexes, whose homology groups are isomorphic to those of the unions of the S_i's (taken at most $\ell + 2$ at a time), as well as computing certain natural homomorphisms between these complexes.

Dealing with small *unions*

Let $P_1, \ldots, P_s$ be homogeneous quadratic polynomials in $R[X_0, \ldots, X_k]$. We denote by

$$P = (P_1, \ldots, P_s) : R^{k+1} \to R^s.$$

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Let

$$\Omega = \{\omega \in R^s \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq s\}.$$

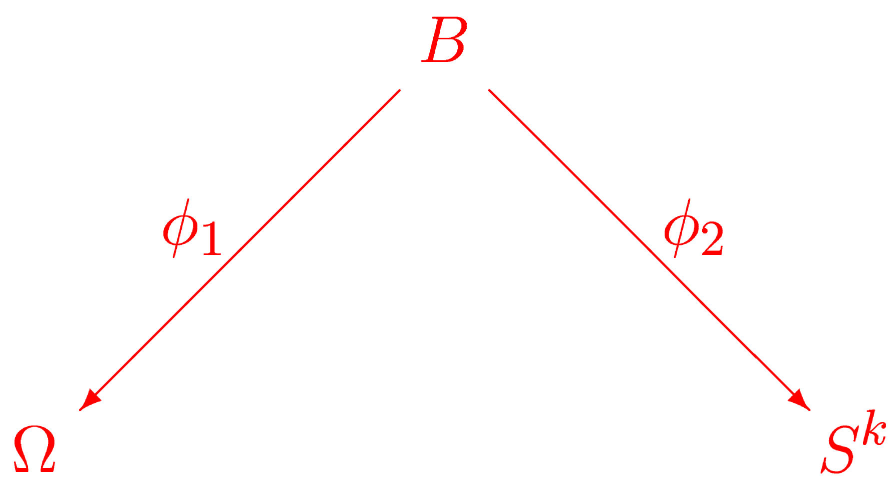
and for $\omega \in \Omega$ let

$$\omega P = \sum_{i=1}^{s} \omega_i P_i.$$

Unions cont.

- Let $B \subset \Omega \times S^k$ be the set defined by,

$$B = \{(\omega, x) \mid \omega \in \Omega, x \in S^k \text{ and } \omega P(x) \geq 0\}.$$



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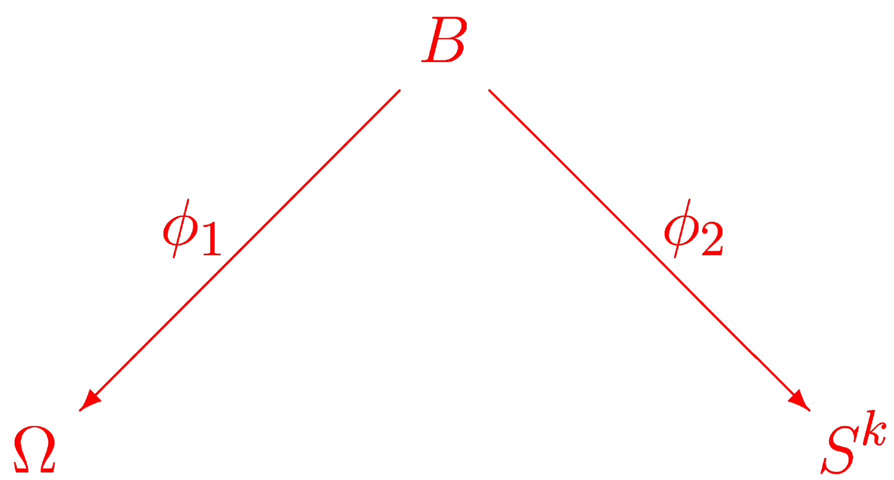

- The map ϕ_2 gives a homotopy equivalence between B and

$$\phi_2(B) = \cup_{i=1}^{s} \{x \in S^k | P_i \leq 0\}$$

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- From this observation we can compute a double complex whose associated to spectral sequence is the Leray spectral sequence of ϕ_1.

And finally ...

On behalf all the participants, a very big THANK YOU to the organizers, for organizing such a great conference !!