Towards a model category for local po-spaces

A framework for a homotopy theory of concurrency

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Outline

1. Introduction - Background, Categorical Framework
2. A basic model category
3. Towards a better model category - via localization

1. Introduction

Concurrency

We would like to understand systems in which processes run concurrently.

Example:

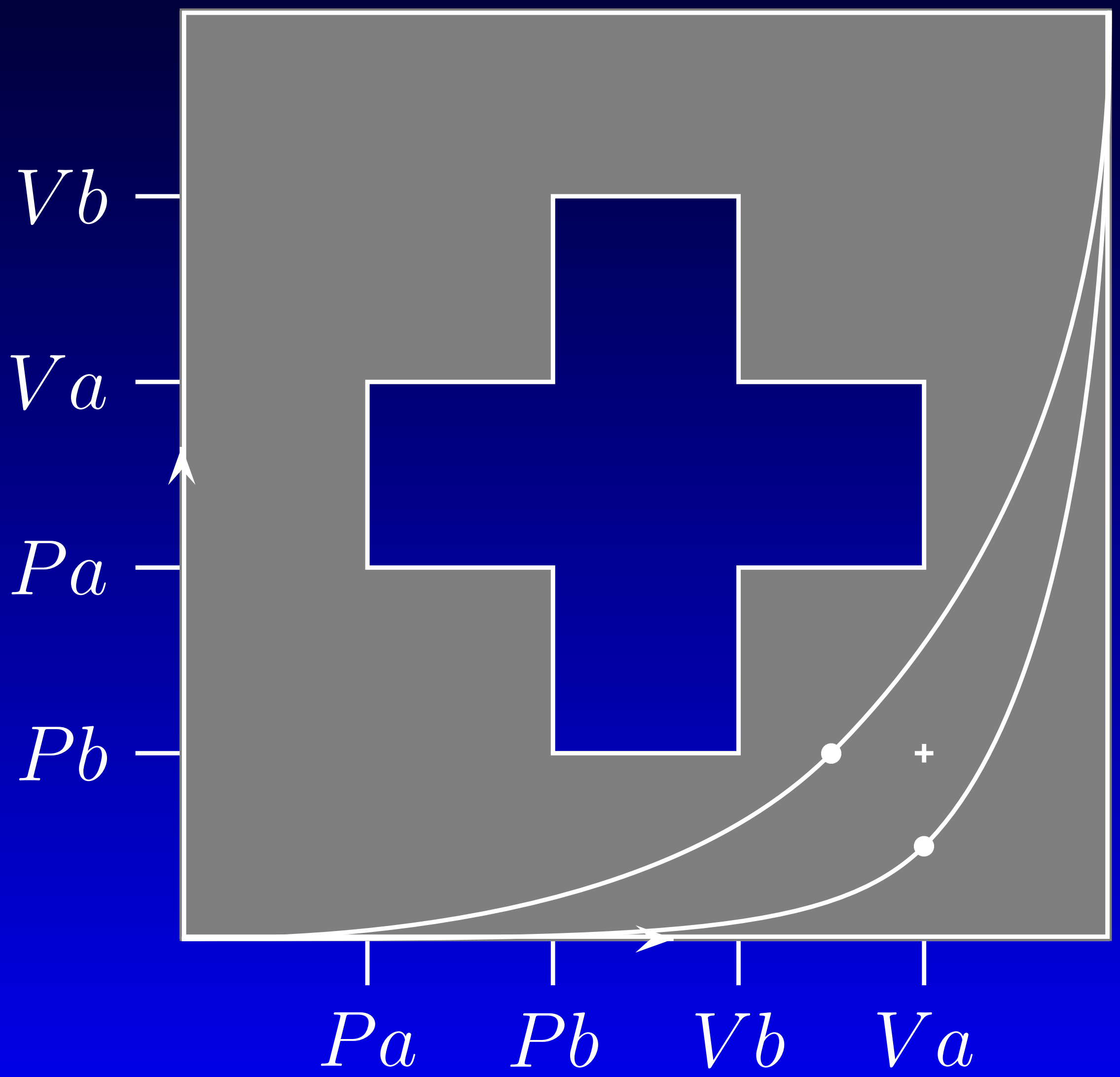
2 processes using 2 shared resources a and b

Notation

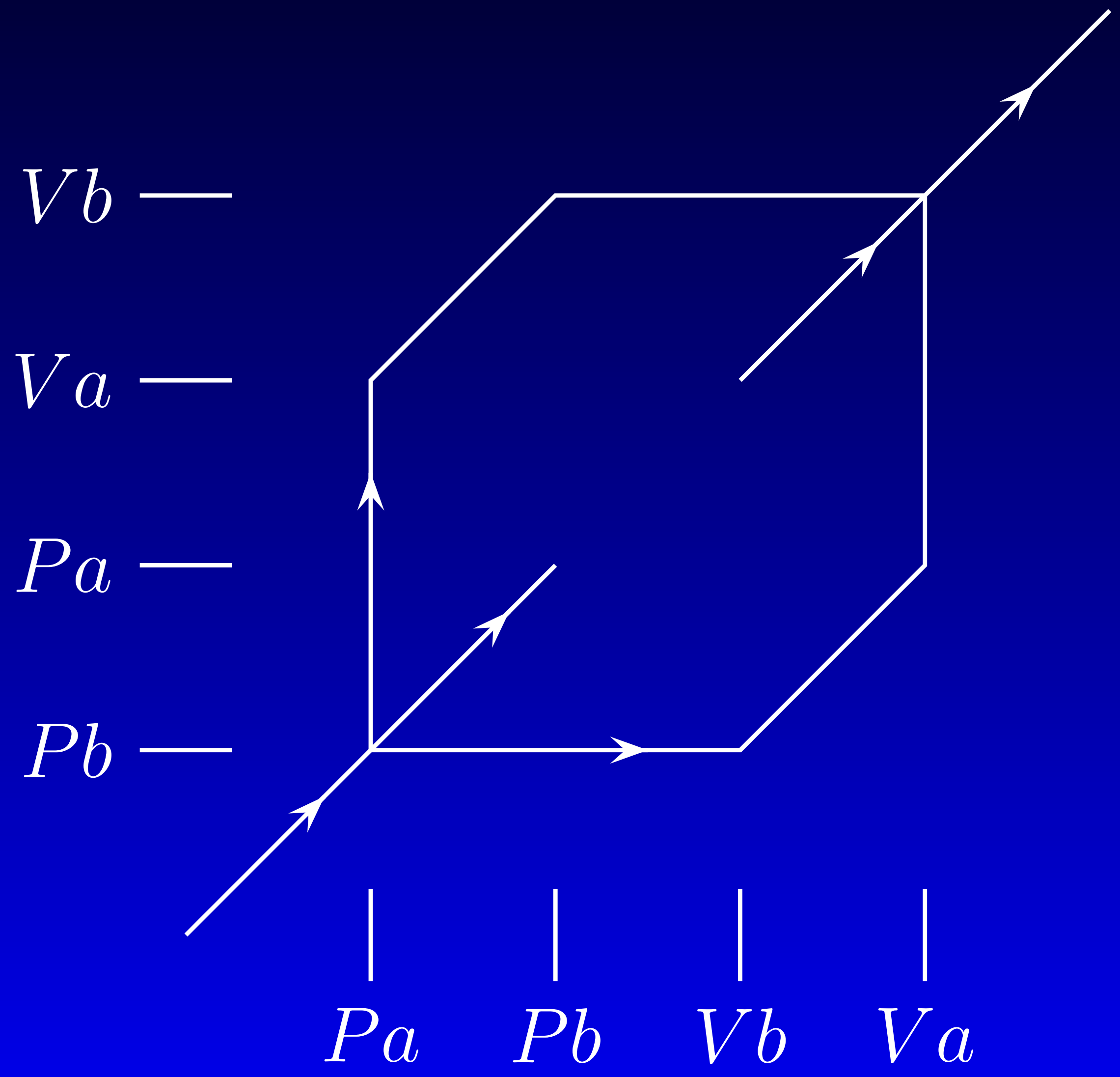
Px - a process locks resource x

Vx - a process releases resource x

The swiss flag



A sub-po-space of the swiss flag



Goal

Develop a framework for concurrency where equivalences are accounted for.

Motivation for studying local po-spaces

Theorem (Fajstrup-Goubault-Raussen): For concurrency, instead of studying HDA/cubical complexes, one can study local po-spaces.

Definitions

Top - a category with
objects: subspaces of $\mathbb{R}^n$ for some n
morphisms: continuous maps

po-space: an object M of **Top** together with a *partial order* (reflexive, transitive, anti-symmetric relation) which is a closed subset of $M \times M$

order atlas: an open cover of po-spaces with compatible partial orders
order atlases are equivalent if they have a common refinement

Definition of **LoPospc**

LoPospc - a category with

objects: $(M, \bar{U})$, $M \in \mathrm{Ob}\,$ **Top**, $\bar{U}$ is an equivalence class of order atlases

morphisms: continuous maps which respect the orders

Products and Subspaces

Remark: There are induced orderings on products and subspaces.

Example:

$$(x, y), (x', y') \in \vec{I} \times \vec{I}, \text{ where } \vec{I} = ([0,1], \leq)$$

$$(x, y) \leq (x', y') \text{ iff } x \leq x' \text{ and } y \leq y'$$

So $(0, \frac{1}{3})$ and $(\frac{2}{3}, 0)$ are not comparable.

Top and LoPospc

Remark: **Top** and **LoPospc** are small categories.

Lemma: There are adjoint functors

$$F : \mathbf{Top} \leftrightarrows \mathbf{LoPospc} : U.$$

Model categories

Recall: A *model category* has 3 special classes of morphisms:

WE, COF, FIB

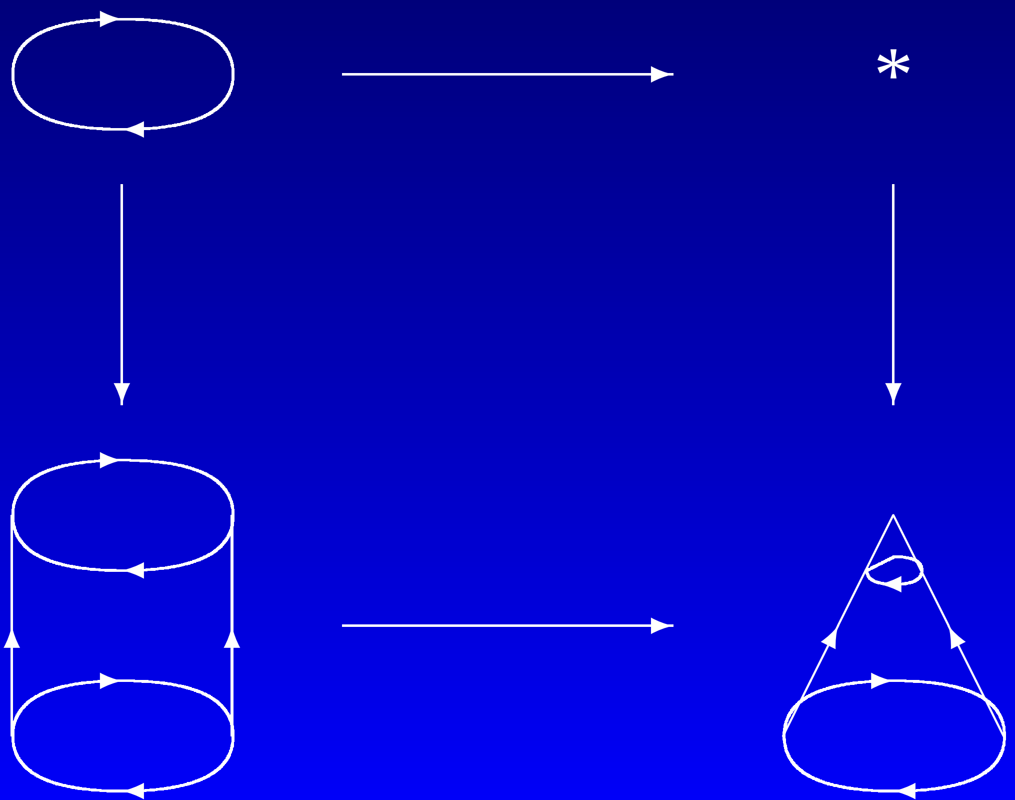
and satisfies a list of axioms (M1,. . .,M5).

Goal: Construct a model category for **LoPospc**.

Problem

Remark: By itself, **LoPospc** cannot have a model structure since it does not contain pushouts.

Example: The pushout of $\vec{S} \times \vec{I}$ by collapsing top is not a po-space



Solution

Need to enlarge the category $\mathbf{LoPospc}$.

Enlarging a Category

Let $\mathbf{C}$ be a small category.
$\mathrm{Pre}(\mathbf{C}) = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ called the presheaves on $\mathbf{C}$

Remark: $\mathbf{C}$ embeds into $\mathrm{Pre}(\mathbf{C})$
(via the Yoneda embedding $y : \mathbf{C} \to \mathrm{Pre}(\mathbf{C})$,
$y(\alpha) = \mathbf{C}(-, \alpha)$)

$\mathrm{sPre}(\mathbf{C}) = \mathbf{sSet}^{\mathbf{C}^{\mathrm{op}}}$ called the simplicial presheaves on $\mathbf{C}$

Remark: $\mathbf{C}$ embeds into $\mathrm{sPre}(\mathbf{C})$
(via the Yoneda embedding $\bar{y} : \mathbf{C} \to \mathrm{sPre}(\mathbf{C})$)

A model structure

Theorem (Jardine): Under condition Q, $\mathrm{sPre}(\mathbf{C})$ has a (proper, simplicial) model structure such that

- COF = monomorphisms, and
- WE = 'stalkwise equivalences'.

Outline for Section 2

- define Q
- show **LoPospc** satisfies Q
- define stalkwise equivalence
- for $\varphi \in \mathrm{Mor}$ **LoPospc** determine when $\bar{y}(\varphi)$ is a stalkwise equivalence

2. A basic model category

C satisfies Q if

$Q1$ **C** is a *site*, and

$Q2$ $\mathrm{Shv}(\mathbf{C})$ has *enough points*

$Q1$

A *Grothendieck topology* J assigns to each $M \in \mathrm{Ob}\,\mathbf{C}$ a collection of *covering families* $\{U_i \to M\} \in \mathrm{Mor}\,\mathbf{C}$ satisfying

1. it contains isomorphism
2. a transitivity condition, and
3. a stability condition

Condition $Q1$

Example: **C** = **Top** and $M \in \mathrm{Ob}$ **Top**
Let $J(M) = \{\text{open covers of } M\}$
(actually a basis for a Grothendieck topology)

A *site* is a small category with a Grothendieck topology.

Example: LoPospc is a site. So it satisfies $Q1$.

Condition $Q2$

$Q2$
Given a site $(\mathbf{C}, J)$ a *sheaf* is a presheaf $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for each covering family, each compatible family of elements of P has a unique amalgamation.

Example: $X \in \mathrm{Ob}\,\mathbf{Top}$, $\mathbf{C} = O(X)$ the open subspaces of X
The presheaf of continuous function is a sheaf.
$\mathrm{Shv}(C, J)$ is a subcategory of $\mathrm{Pre}(\mathbf{C})$.

Points

A *point* on $\mathrm{Shv}(\mathbf{C}, J)$ is a pair of adjoint functors

$$p^* : \mathrm{Shv}(\mathbf{C}, J) \leftrightarrows \mathbf{Set} : p_*$$

such that p_* preserves finite limits.

$\mathrm{Shv}(\mathbf{C}, J)$ has *enough points* if given

$$f \neq g : P \to Q \in \mathrm{Shv}(\mathbf{C}, J)$$

then there is a point p^* such that

$$p^* f \neq p^* g : p^* P \to p^* Q \in \mathbf{Set}.$$

Points in LoPospc

Let $\mathbf{C} = \mathbf{LoPospc}$. Let $Z \in \operatorname{Ob} \mathbf{C}$ and let $x \in Z$. Define

$$p_x^* : \operatorname{Pre}(\mathbf{C}) = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}$$

$$F \mapsto \operatorname*{colim}_{x \in L^{\text{open}} \subseteq Z} F(L)$$

$$\begin{array}{ccc} \operatorname{Pre}(\mathbf{C}) & \xrightarrow{p_x^*} & \mathbf{Set} \\ a \downarrow \uparrow i & \nearrow_{p_x^*} & \\ \operatorname{Shv}(\mathbf{C}) & & \end{array}$$

Proposition (B): p_x^* descends to a point on $\operatorname{Shv}(\mathbf{C})$.

Points in LoPospc (continued)

Theorem (B): These points provide enough points for $\mathrm{Shv}(\mathbf{C})$.

So **LoPospc** satisfies $Q2$.

Stalks

Let p^* be a point in $\mathrm{Shv}(\mathbf{LoPospc})$.
Let $\alpha \in \mathrm{sPre}\,\mathbf{LoPospc} = \mathbf{sSet}^{\mathbf{LoPospc}^{\mathrm{op}}}$.
The *stalk* of α at p^* is given by

$$\alpha_p = \{p^* a(\alpha_n)\}_{n \geq 0}$$

Remark: $\alpha_n \in \mathrm{Pre}(\mathbf{LoPospc})$,
$a(\alpha_n) \in \mathrm{Shv}(\mathbf{LoPospc})$, $p^* a(\alpha_n) \in \mathbf{Set}$.

Say that $f : P \to Q \in \mathrm{sPre}\,\mathbf{LoPospc}$ is a *stalkwise equivalence* if for all points $p^* \in \mathrm{Shv}(\mathbf{LoPospc})$, $f_p : P_p \to Q_p \in \mathbf{sSet}$ is a weak equivalence.

Stalkwise equiv. in LoPospc

Recall: Let $\varphi : X \to Y \in \mathrm{Mor}\,$ **LoPospc**.
Then $\bar{y}(\varphi) : \bar{y}(X) \to \bar{y}(Y) \in \mathrm{Mor}\,\mathrm{sPre}($**LoPospc**$)$.

Theorem (B): $\bar{y}(\varphi)$ is a stalkwise equivalence if and only if φ is an isomorphism.

Summary: **LoPospc** includes into a model category such that

- COF $\cap$**LoPospc** $=$ { monomorphisms }
- WE $\cap$**LoPospc** $=$ { isomorphisms }

3. Localization

We need to introduce non-trivial equivalences.

Example: For **Top** one localizes with respect to the maps

$$X \times I \xrightarrow{\text{proj}} X,$$

where I is the unit interval $[0, 1]$.

Let $\mathcal{I} = \{\bar{y}(X \times I \to X) | X \in \mathrm{Ob}\,\textbf{Top}\}$.

Then $\mathrm{sPre}(\textbf{Top})/\mathcal{I}$ produces the usual homotopy theory on **Top**.

Localization for **LoPospc**

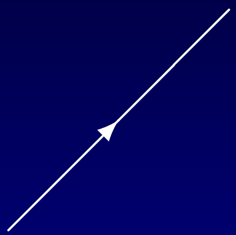
Question: What is a good $\mathcal{I}$ for **LoPospc**?

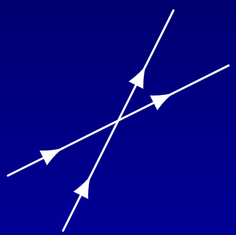
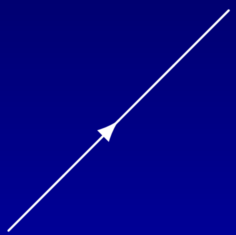
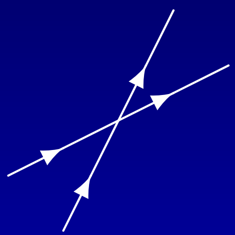
Ideas:

- $\mathcal{I}$ as above
- $\mathcal{I}$ as above but with $\vec{I}$ instead of $\mathcal{I}$
- dihomotopy equivalences
- d-homotopy equivalence

Problems

With each of these we have the following weak

equivalences:  $\xrightarrow{\sim} *$ and

 $\xrightarrow{\sim} *$ but  and 

are not the same in concurrency.

Recall: The model structure on $\mathrm{sPre}\,\mathbf{LoPospc}$ is proper.

Thus so is the model structure on $\mathrm{sPre}(\mathbf{LoPospc})/\mathcal{I}$.

Left proper

In a left proper model category the pushout of a weak equivalence over a cofibration is a weak equivalence.

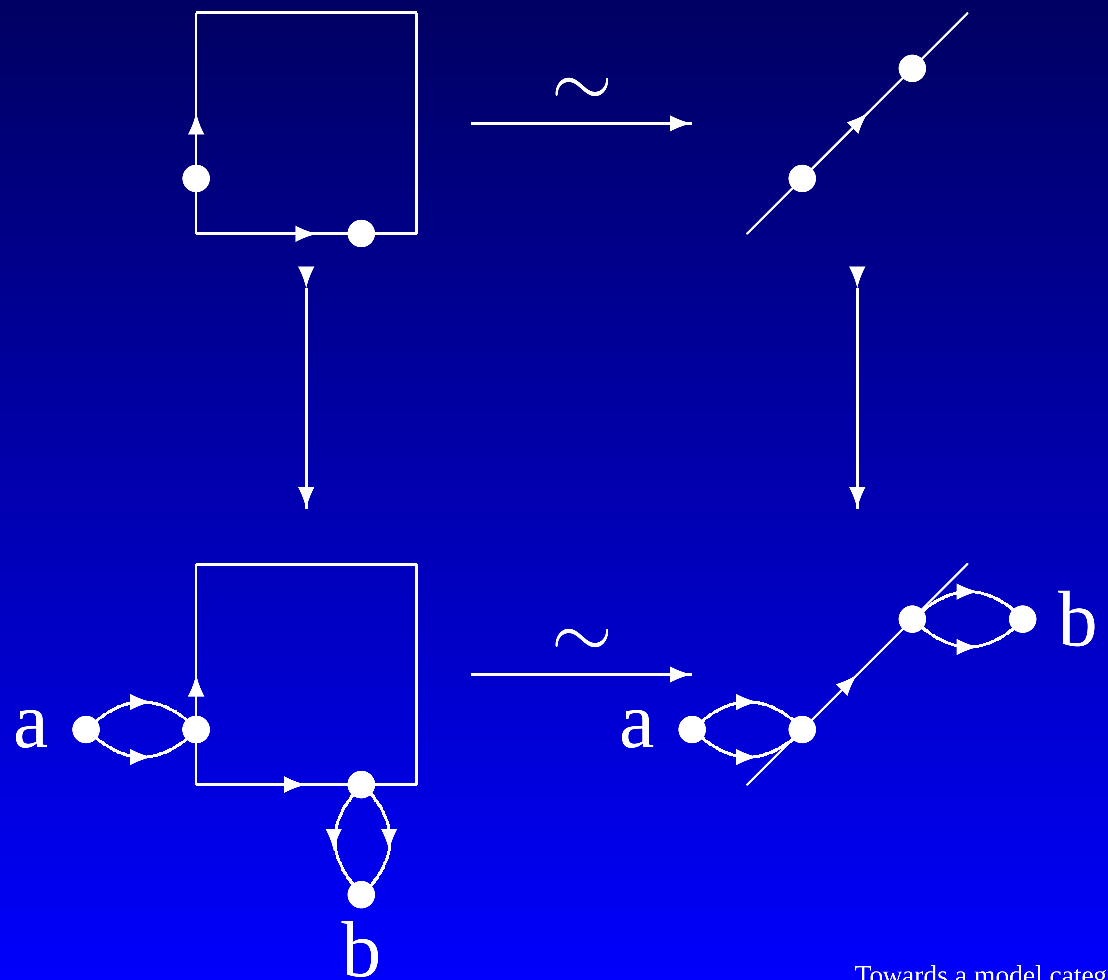
$$\begin{array}{ccc} A & \xrightarrow{\simeq} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\simeq} & Y \end{array}$$

Thus $A \to X \in \mathcal{I}$ and $A \to B \in$ COF implies that $B \to Y \in$ WE in $\mathrm{sPre}(\mathbf{LoPospc})/\mathcal{I}$.

Problem

Assume some map $\vec{I} \times \vec{I} \to \vec{I} \in \mathcal{I}$.

Choose two points $x, y \in \vec{I} \times \vec{I}$ such that $x \not\leq y$ and $y \not\leq x$.



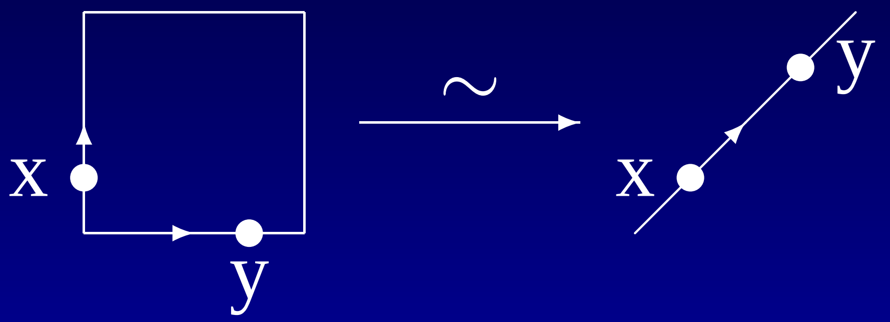
Possible Solution

Use *context*.
Work with $\mathbf{A \downarrow LoPospc}$ instead of $\mathbf{LoPospc}$, where A depends on the pastings one wants to consider.

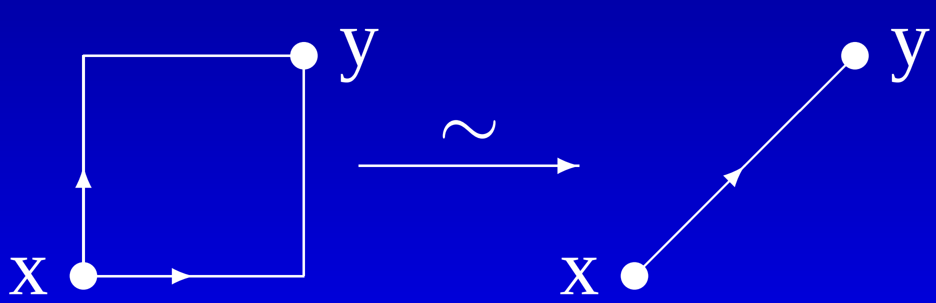
Example

$A = \{x, y\}$.
Then **A ↓ LoPospc** is the category of local po-spaces with two marked points.

Furthermore,



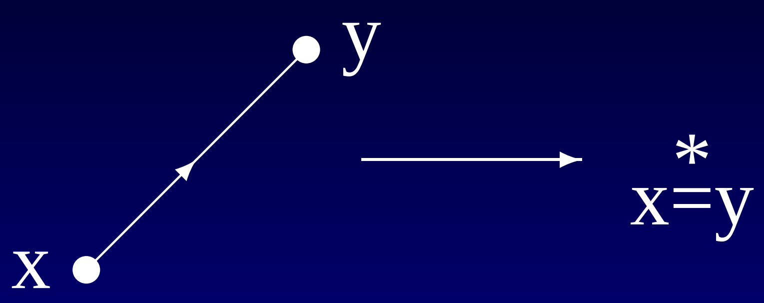
is not a dihomotopy equivalence, but



is.

Another example

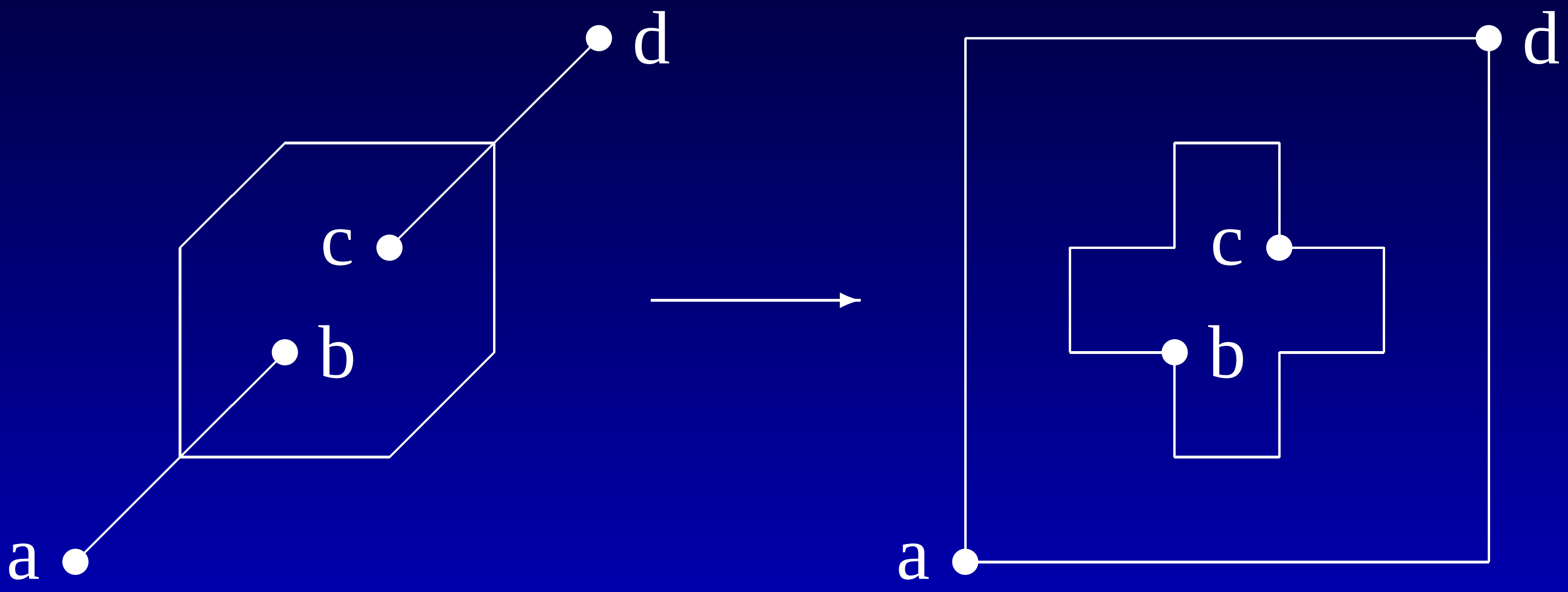
Also



is not a dihomotopy equivalence.

Context for the Swiss flag

Let $A = \{a, b, c, d\}$. Then



is a d-homotopy equivalence in $\mathbf{A} \downarrow \mathbf{LoPospc}$.

Proposal

Work with

$$\bar{y}(A) \downarrow \mathrm{sPre}(\mathbf{LoPospc})$$

which inherits a model structure from $\mathrm{sPre}(\mathbf{LoPospc})$. Then localize

$$(\bar{y}(A) \downarrow \mathrm{sPre}(\mathbf{LoPospc}))/\mathcal{I}$$

where

$$\mathcal{I} = \{\bar{y}(\text{ d-homotopy equivalences})\}$$

or

$$\mathcal{I} = \{\bar{y}(\text{ dihomotopy equivalences})\}.$$