Higher order automata, cubical sets, and some conjectures of Grothendieck

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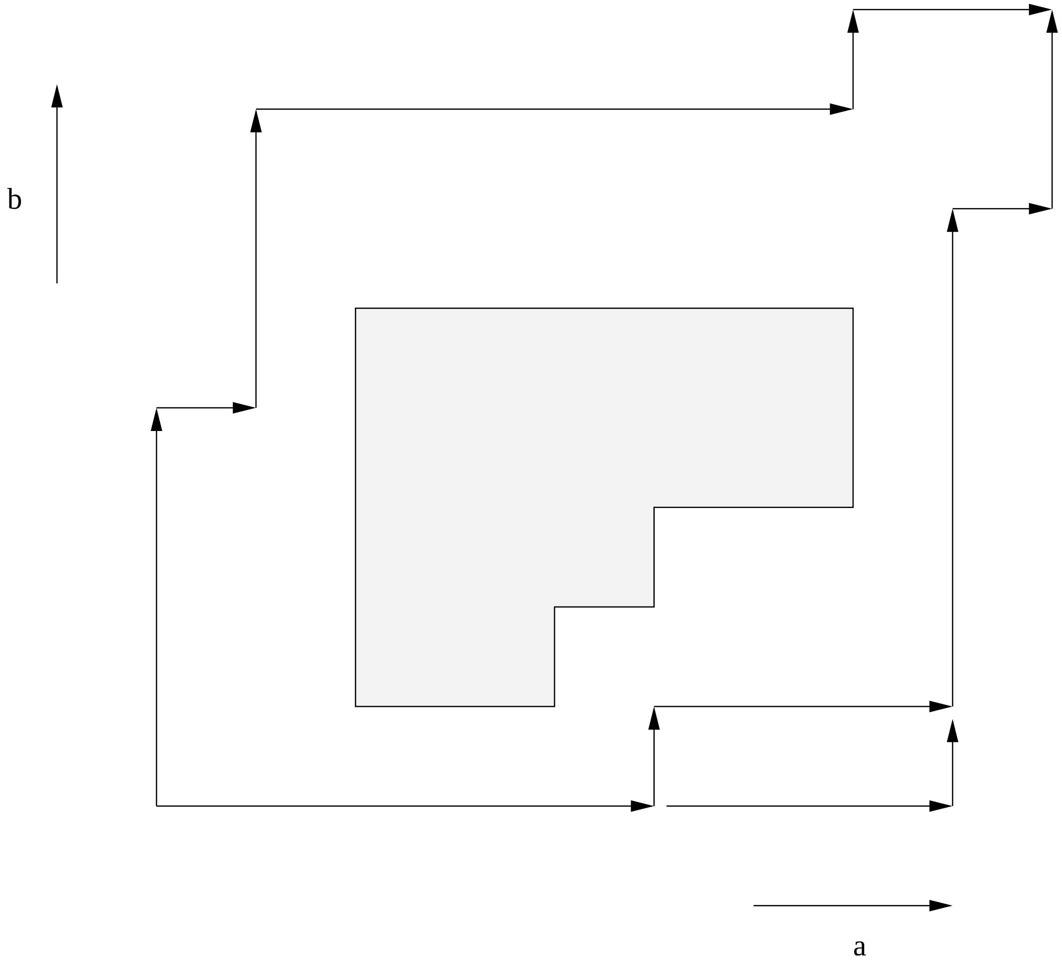
Higher order automata



V. Pratt, “Modelling Concurrency with Geometry”, 1990.

What's going on:

- Hidden lattice of states (vertices) and squares outside of shaded area: tiled floor with tiles corresponding to shaded area removed, ie. a complex X of 2-cubes.
- Each "square" outside of shaded area models independent action of a and b. Constraints are modelled by the missing shaded area (eg. shared resources). Concurrent behaviour of system corr. to maps in "path category" of X — coarse obstructions in fund. groupoid.
- More processors involve more dimensions, and corr. constraints are higher dimensional holes.

A *higher dimensional automaton* Y is a cubical complex, and behaviour of the system is reflected in homotopy invariants.

Homotopy types

How to study the homotopy invariants of a cubical complex X?

- Represent X as a topological space, and compute its homotopy type.
- First combinatorial method: represent X as a simplicial complex (“triangulation” of X) and compute its homotopy type in the category of simplicial sets.
- Second combinatorial method: compute the homotopy type of X in an internal homotopy theory for cubical sets.

Simplicial sets

Old-time definition: A *simplicial set* X consists of sets X_n, $n \geq 0$, together with functions $d_i : X_n \to X_{n-1}$ (face maps), $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$, satisfying simplicial identities:

$$d_i d_j = d_{j-1} d_i \quad i < j,$$

$$s_i s_j = s_{j+1} s_i \quad i \leq j,$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ 1 & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

$\mathbf{\Delta}$ = *ordinal number category* of all finite sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and all order-preserving functions $\mathbf{n} \to \mathbf{m}$.
Modern definition: A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$.

Simplicial set map $=$ natural transformation.
$s\mathbf{Set} =$ category of simplicial sets.

Examples:

- $\Delta^n = \hom_{\mathbf{\Delta}}(\ , \mathbf{n})$, has classifying n-simplex $\iota_n = 1_{\mathbf{n}}$
- $\partial\Delta^n =$ subcomplex of Δ^n generated by all faces $d_i\iota_n = d^i : \mathbf{n-1} \to \mathbf{n}$ (boundary)
- $\Lambda^n_k =$ subcomplex generated by $d_i\iota_n$, $i \neq k$ (k^{th} horn).
- A (finite oriented) simplicial complex K is a subcomplex of some Δ^n.
- $C =$ small category. BC is "nerve" or "classifying object" of C. $BC_n = \hom_{cat}(\mathbf{n}, C) =$ strings of arrows of length n in C. Example: $\Delta^n = B\mathbf{n}$.

Realization

$|\Delta^n| = \{(x_0, \ldots, x_n) \mid \sum x_i = 1, x_i \geq 0\}$. $\mathbf{n} \mapsto |\Delta^n|$ is functorial in $\mathbf{n}$.

The *simplex category* $\boldsymbol{\Delta}/X$ of a simp set X has objects all maps $\Delta^n \to X$ and morphisms all comm triangles

$$\begin{array}{ccc} \Delta^n & \longrightarrow & \Delta^m \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Realization $|\ | : s\mathbf{Set} \to \mathbf{Top}$: $|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|$.

Singular functor $S : \mathbf{Top} \to s\mathbf{Set}$: $S(Y)_n = \hom(|\Delta^n|, Y)$.

Realization is left adjoint to the singular functor, ie. X is a colimit of its simplices.

These functors determine an “equivalence of homotopy categories”.

Closed model categories

A *closed model category* $\mathcal{M}$ is a category which is equipped with three classes of morphisms, called *cofibrations*, *fibrations* and *weak equivalences* which together satisfy the following axioms:

CM1: The category $\mathcal{M}$ is closed under all finite limits and colimits.

CM2: Suppose that the following diagram commutes in $\mathcal{M}$:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & {}_{h}\searrow \quad \swarrow_{f} & \\ & Z & \end{array}$$

If any two of f, g and h are weak equivs, so is the third.

CM3: If f is a retract of g and g is a weak equiv, fibn or cof, so is f.

CM4: Suppose given

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle p} \\ V & \longrightarrow & Y \end{array}$$

where i is a cofibration and p is a fibration. Then the filler (dotted) arrow exists if either i or p is also a weak equiv.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibn and i is a triv cof,

(b) $f = q \cdot j$ where q is a triv fibn and j is a cof.

Examples:

- Chain complexes: weak equivalences are homology isomorphisms, $p : C \to D$ is fibration iff $C_n \to D_n$ is surjective for $n > 0$. Chain complexes of projectives are the cofibrant objects.
- Compactly generated Hausdorff spaces: weak equivalences are weak homotopy equivalences, fibrations are Serre fibrations. CW-complexes are cofibrant.
- Simplicial sets: cofibrations are monomorphisms, weak equivalences are simp. set maps $X \to Y$ which induce homotopy equivalences $|X| \to |Y|$, fibrations are Kan fibrations.

Fibrations

A simplicial set map $p : X \to Y$ is a *Kan fibration* if the dotted arrow exists making the diagram commute in every dia.

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow^{\cdots} & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

ie. p has the *right lifting property* wrt. all inclusions $\Lambda^n_k \subset \Delta^n$.

Hard results (consequences of simplicial approximation):

- $p : X \to Y$ is a fibration and a weak equivalence iff p has the right lifting property wrt all inclusions $\partial\Delta^n \subset \Delta^n$.
- The realization of a Kan fibration is a Serre fibration.
- There is an equivalence $\mathrm{Ho}(s\mathbf{Set}) \simeq \mathrm{Ho}(\mathbf{Top})$.

Trivial fibrations

An explanation: $p : X \to Y$ is a fibration and a weak equivalence iff the lifting exists in all diagrams

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

- every $(n-1)$-dim homotopy element which maps to the trivial element in Y must be trivial in X
- all n-dim homotopy elements of Y lift to n-dim homotopy elements of X
- p has a section $\sigma : Y \to X$ which can be constructed cell by cell and there is a homotopy $X \times \Delta^1 \to X$: $\sigma \cdot p \simeq 1_X$

The box category

$\boldsymbol{\Delta}$ is a category of models for simplicial sets. Corr category for cubical sets is the *box category* $\square$.

Objects of $\square$: posets $\mathbf{1}^{\times n}$, $n \geq 1$, (combinatorial hypercubes)

$\mathbf{1}^{\times n} = \{(\epsilon_1, \ldots, \epsilon_n) | \, \epsilon_i = 0, 1\} = \mathcal{P}(\underline{n})$ where $\underline{n} = \{1, \ldots, n\}$.

For $A \subset B \subset \underline{n}$, interval

$$[A, B] = \{C | A \subset C \subset B\} \cong \mathcal{P}(B - A),$$

and defines *coface* $d : \mathcal{P}(\underline{k}) \cong \mathcal{P}(B - A) \subset \mathcal{P}(\underline{n})$

$B \subset \underline{n}$: $C \mapsto C \cap B$ determines *codegeneracy*
$s : \mathcal{P}(\underline{n}) \to \mathcal{P}(B) \cong \mathcal{P}(\underline{m})$

Morphisms of $\square$ are the functors (poset maps) $\mathbf{1}^{\times n} \to \mathbf{1}^{\times m}$ generated by cofaces and codegeneracies.

Cubical sets

A *cubical set* is a functor $X : \Box^{op} \to \mathbf{Set}$. Morphisms of cubical sets are natural transformations. $\Box - \mathbf{Set}$ is category of cubical sets. $X_n = X(\mathbf{1}^{\times n})$ are n-cells of X.
Examples:

- $\Box^n = \hom_{\Box}(\ , \mathbf{1}^{\times n})$, standard n-cell, has generator ι_n.
- For $1 \leq i \leq n$ there are two cofaces $d^{i,0} = [\emptyset, \underline{n} - \{i\}]$ and $d^{i,1} = [\{i\}, \underline{n}]$.

  $\partial\Box^n \subset \Box^n$ is generated by all $d_{i,\epsilon}(\iota_n)$ (cubical complex).
- There is subobject $\sqcap^n_{i,\epsilon} \subset \Box^n$ gen by all faces $d_{j,\nu}(\iota_n)$, $(j, \nu) \neq (i, \epsilon)$.
- $C =$ small category: *cubical nerve* $B_{\Box}(C)$ has $B_{\Box}(C)_n = \hom(\mathbf{1}^{\times n}, C)$, cubical structure def by precomp with box morphisms. **Warning**: $B_{\Box}(\mathbf{1}^{\times n}) \neq \Box^n$.

Triangulation

$X =$ cubical set: *cell category* $\square/X$ has as objects all cells $\square^n \to X$, and for morphisms all commutative triangles

$$\begin{array}{ccc} \square^n & \longrightarrow & \square^m \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Triangulation functor $|\ | : \square - \mathbf{Set} \to s\mathbf{Set}$ def by

$$|X| = \varinjlim_{\square^n \to X} B(\mathbf{1}^{\times n}).$$

Define *cubical singular functor* $S_\square : s\mathbf{Set} \to \square - \mathbf{Set}$ by

$$S_\square(Y)_n = \hom(B(\mathbf{1}^{\times n}), Y).$$

Triangulation functor is left adjoint to cubical singular functor.

Cubical homotopy theory

A map $f : X \to Y$ of cubical sets is a *weak equivalence* if $|X| \to |Y|$ is a weak equivalence of simp. sets. A *cofibration* of cubical sets is an inclusion. Fibrations are def. by a right lifting property wrt all trivial cofibrations.

Theorem: (Jardine, Cisinski)

1) With these definitions, $\square - \mathbf{Set}$ satisfies the axioms for a closed model category.

2) Triangulation and cubical singular functors induce an equivalence $\mathrm{Ho}(\square - \mathbf{Set}) \simeq \mathrm{Ho}(s\mathbf{Set})$.

Theorem: (Cisinski)
A map $p : X \to Y$ of cubical sets is a fibration if and only if it has the RLP with respect to all inclusions $\sqcap^n_{i,\epsilon} \subset \square^n$.

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$\mathcal{A}$-sets

Contravariant functors on $\mathbf{\Delta}$ and $\square$ give combinatorial models for homotopy theory.
For what other small categories $\mathcal{A}$ does the category of $\mathcal{A}$-sets ie. functors $X : \mathcal{A}^{op} \to \mathbf{Set}$ have this property?

Each $a \in \mathcal{A}$ determines a standard a-cell $\Delta^a = \hom_{\mathcal{A}}(\ ,a)$.

The *cell category* $i_{\mathcal{A}}X$ of an $\mathcal{A}$-set X has objects given by all a-cells $\Delta^a \to X$ and morphisms

$$\begin{array}{ccc} \Delta^a & \longrightarrow & \Delta^b \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The functor $X \mapsto i_{\mathcal{A}}X$ has a right adjoint $C \mapsto i_{\mathcal{A}}^*(C)$, where $i_{\mathcal{A}}^*(C)_a = \hom(\mathcal{A}/a, C)$.

Test categories

Defn: A small category $\mathcal{A}$ is a *test category* if

1) $B\mathcal{A}$ is a contractible simplicial set
2) The map $\alpha : i_{\mathcal{A}} i_{\mathcal{A}}^*(C) \to \mathcal{A}$ is aspherical ($Bi_{\mathcal{A}}(i_{\mathcal{A}}^*(C) \times \Delta^a) \simeq *$) if C has a terminal object.

Examples of test categories:

- $\mathbf{\Delta}$: $i_{\mathbf{\Delta}}^*$ takes natural transformations to homotopies, since the last vertex map $\mathbf{\Delta}/\mathbf{1} \to \mathbf{1}$ defines $\Delta^1 \to i_{\mathbf{\Delta}}^*(\mathbf{1})$.
- $\square$: show that the functor $i_{\square} B_{\square}(C) \to \square$ is aspherical if C has a terminal object
- If $\mathcal{A}$ and $\mathcal{B}$ are test categories, so is $\mathcal{A} \times \mathcal{B}$ (eg. bisimplicial, bicubical, cubical-simplicial sets are described by test categories).

Homotopy theories of $\mathcal{A}$-sets

Say that a map $f : X \to Y$ is a weak equivalence if the map $B(i_{\mathcal{A}}X) \to B(i_{\mathcal{A}}Y)$ is a weak equivalence of simplicial sets. A cofibration is a monomorphism.

Theorem: (Cisinski)

1) Suppose that $\mathcal{A}$ is a test category. Then with the definitions above the category of $\mathcal{A}$-sets satisfies the axioms for a closed model category.

2) The functors

$$i_{\mathbf{\Delta}}^* i_{\mathcal{A}} : \mathcal{A} - \mathbf{Set} \leftrightarrows s\mathbf{Set} : i_{\mathcal{A}}^* i_{\mathbf{\Delta}}$$

induce an equivalence $\mathrm{Ho}(\mathcal{A} - \mathbf{Set}) \simeq \mathrm{Ho}(s\mathbf{Set})$.

Comments:

- The model structure for cubical sets is a corollary.
- Proof uses a little surprise: $X \mapsto Bi_{\mathcal{A}}(X)$ takes pushouts of cofibrations to homotopy co-cartesian diagrams of simplicial sets.
- The test category assumption is primarily used for the equiv. of homotopy categories (which is formal).
- The statement and proof generalize to all localizations of the category of simplicial sets: formally inverting a set of simp. set maps induces a model structure on $\mathcal{A}$-sets, giving equiv. homotopy categories.

Weak equivalence classes

Cisinski's fibration theorem for cubical sets involves Grothendieck's theory of weak equivalence classes ("fundamental localisers").

A *weak equivalence class* is a class $\mathcal{W}$ of functors between small categories s.t. the following hold:

1) Every identity is in $\mathcal{W}$.
2) (**CM2**) Given $C \xrightarrow{f} D \xrightarrow{g} E$, if any two of f, g, gf are in $\mathcal{W}$, so is the third.
3) Given $A \xrightarrow{i} B \xrightarrow{r} A$ with $ri = 1$, if $ir \in \mathcal{W}$ then $r \in \mathcal{W}$.
4) Given a commutative triangle of functors $\begin{array}{ccc} A & \xrightarrow{u} & B \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & C & \end{array}$ if all $\alpha/c \to \beta/c$ are in $\mathcal{W}$ then $u \in \mathcal{W}$.

Grothendieck's conjectures

Example: $\mathcal{W}_\infty =$ class of all functors $C \to D$ such that $BC \to BD$ is a weak equiv. of simplicial sets.

Conjecture 1: $\mathcal{W}_\infty$ is the smallest weak equivalence class.

Conjecture 2: Suppose that $\mathcal{W}$ is a weak equiv. class and $\mathcal{A}$ is a test category. Then the class of all maps $X \to Y$ of $\mathcal{A}$-sets such that $i_{\mathcal{A}}X \to i_{\mathcal{A}}Y$ is in $\mathcal{W}$ is the class of weak equivalences for a model structure on $\mathcal{A}$-sets with cofibrations $=$ monomorphisms.

Cisinski proved Conjecture 1 in its entirety, and Conjecture 2 in the "accessible" case (ie. localized version of his theorem above).

Conjecture 1: show that if $A \to B$ is a trivial cofibration of simplicial sets, then the functor $i_{\mathbf{\Delta}}A \to i_{\mathbf{\Delta}}B$ is in $\mathcal{W}$.

Epilogue

Cisinski's theorem and its localized version are special cases of results for presheaves of $\mathcal{A}$-sets on an arbitrary small Grothendieck site $\mathcal{C}$.

Example (speaking in tongues): this gives motivic homotopy theories for each category of presheaves of $\mathcal{A}$-sets on the smooth Nisnevich site for a scheme, all of which are equivalent to the Morel-Voevodsky model.

Possible example: Suppose one has multiple parallel processor machines on a network, all of which have their own internal logical structures. Suppose also that there is some global rule set for the network, specified by a Grothendieck topology, some localization procedure, or both. Such a system could possibly be modelled by a presheaf of cubical sets, with homotopy type giving local to global information.

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