Complexity and Tractability issues in Topological aspects of 3-D Computational Electromagnetics

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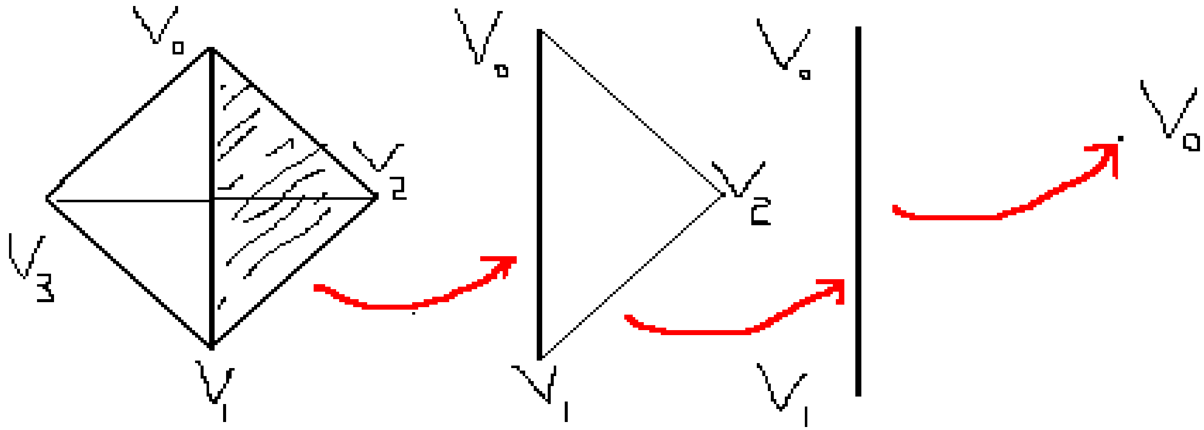
Outline

1. Data structures and Computational Electromagnetics.
   - Chains, Cochains, and (co)homology.
2. What is Algebraic Topology and what is computable?
3. Dimensionality and issues of computability and decidability
4. Issues of Computability and decidability in 3-D
5. Conclusions

Simplicial (Chain) Complexes



$$\sigma_p = \langle V_0, \ldots, V_p \rangle \qquad \textit{(p-simplex)}$$

$$\partial_p \sigma_p = \partial\left(\langle V_0, \ldots, V_p \rangle\right) = \sum_{i=0}^{p} (-1)^i \langle V_0, \ldots, V_{i-1}, V_{i+1}, \ldots, V_p \rangle$$

C_p=formal linear combinations of simplices,

$\partial_p : C_p \to C_{p-1}$ and for all p, $\partial_{p-1}\partial_p = 0$

$$\int_\Omega \omega = \langle \omega, \Omega \rangle : C^p \times C_p \to \text{Real numbers (or integers)}$$

↑ cochains ↑ chains

Let's Slow Down

$$\left\langle d\omega, \Omega \right\rangle = \int_{\Omega} d\omega = \int_{\partial\Omega} \omega = \left\langle \omega, d\Omega \right\rangle \quad \Rightarrow \quad d = \partial^{T}$$

$$0 \rightarrow C_n(K) \xrightarrow{\partial n} \ldots \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \ldots \rightarrow C_1(K) \xrightarrow{\partial_0} C_0(K) \rightarrow 0$$

$$0 \leftarrow C^n(K) \xleftarrow[d^{n-1}=\partial_n^T]{} \ldots \xleftarrow[d^{p-1}=\partial_p^T]{} C^{p-1}(K) \xleftarrow[d^{p-2}]{} \ldots \xleftarrow[d^1]{} C^1(K) \leftarrow C^0(K) \leftarrow 0$$

and for differential forms:

$$0 \leftarrow \Omega^n(M) \xleftarrow[d]{} \ldots \leftarrow \Omega^{p-1}(M) \leftarrow \ldots \leftarrow \Omega^1(M) \leftarrow \Omega^0(M) \leftarrow 0$$

where K is a triangulation of M

de Rham Map:

$$R : \Omega^p(M) \rightarrow C^p(K)$$

$$\omega \; \alpha \; \int_\sigma \omega$$

Whitney Map:

$$W : C^p(K) \rightarrow \Omega^p(M)$$

$$\sigma^p \; \alpha \;\; p! \sum_{k=0}^{p} (-1)^k \mu_{i_k} d\mu_{i_0} \wedge \ldots \wedge \hat{d}\mu_{i_k} \wedge \ldots \wedge d\mu_{i_p}, \;\; p > 0$$

Facts:

$$RW = Identity \;\; on \, C^p(K),$$

$$"WR \rightarrow Identity" \;\; on \, \Omega^p(M)$$

(Co)homology

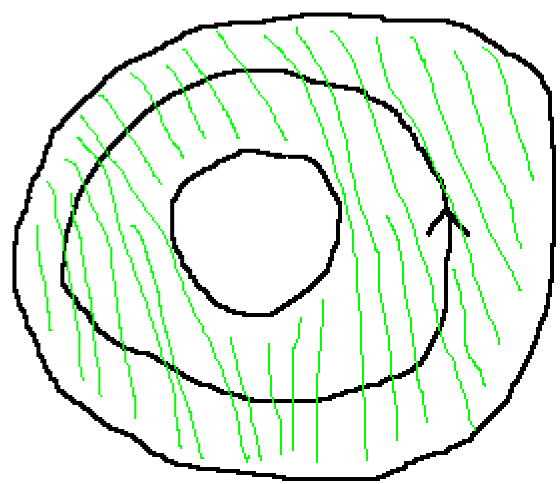
$$d^p d^{p-1} = 0, \partial^p \partial^{p+1} = 0$$

$$H^p(K) = \frac{Kernel\ d^p}{Image\ d^{p-1}}$$

$$e.g.\begin{cases} curl\ X = 0 \\ X' \sim X \quad \text{if } X' = X + grad\ \phi \end{cases}$$

$$H_p(K) = \frac{Kernel\ \partial^p}{Image\ \partial_{p+1}}$$

$$e.g.\begin{cases} \partial c = 0 \\ c' \sim c \quad \text{if } c' = c + \partial b \end{cases}$$



Some Dates and Observations:

•deRham(1931) $H^p(K) \simeq H_p(K)$ (coefficients in a field)

•Isomorphisms e.g. $H^p(K) \simeq H^p_{deR}(M)$

Isomorphisms are automatic if the “theory” satisfies the “Eilenberg-Steenrod Axioms” (~1940’s)

•The W (Whitney) map was devised by André Weil in 1952. He wanted to see H^* □ H_* on the (co)chain level.

•Note: none of the above requires a metric or a Hodge star. Can be implemented using integer arithmetic or finite fields.

i.e. Z, or Z/pZ=Z_p

•R&W reconcile (co)homological aspects with FEA.

This is the <u>ideal set-up for M.E.</u>

$$\int_{\partial S} \overline{\mathrm{E}} \cdot d\overline{\lambda} = -\frac{d}{dt}\int_{S} \overline{B} \cdot d\overline{S} \qquad \int_{\partial V} \overline{B} \cdot d\overline{S} = 0$$

$$\int_{\partial S'} \overline{H} \cdot d\overline{\lambda} = \int_{S'} \overline{J} \cdot d\overline{S} + \frac{d}{dt}\int_{S'} \overline{D} \cdot d\overline{S} \qquad \int_{\partial V'} \overline{D} \cdot d\overline{S} = \int_{V'} \rho \, dV$$

~~$\overline{D} = \varepsilon \overline{E}$~~ ~~$\overline{B} = \mu \overline{H}$~~

Quasistatics in M, ∂M interfaces with a perfect conductor

$$V = \int_c \overline{E} \cdot d\overline{\lambda}, \qquad \left[\overline{\mathrm{E}} \cdot \mathrm{d}\overline{\lambda}\right] \in H^1(M, \partial M; Z),$$

$$\left[\mathrm{c}\right] \in H_1\left(M, \partial M; Z\right)$$

$$Q = \oint_s \overline{D} \cdot d\overline{S}, \qquad \left[\overline{\mathrm{D}} \cdot \mathrm{d}\overline{S}\right] \in H^2(M; Z),$$

$$\left[\mathrm{S}\right] \in H_2\left(M; Z\right)$$

$$\mathrm{I} = \oint_{c'} \overline{H} \cdot d\overline{\lambda}, \qquad \left[H \cdot \mathrm{d}\lambda\right] \in H^1(M; Z),$$

$$\left[\mathrm{c}'\right] \in H_1\left(M; Z\right)$$

$$\Phi = \int_{s'} \overline{B} \cdot d\overline{S}, \qquad \left[\overline{\mathrm{B}} \cdot \mathrm{d}\overline{S}\right] \in H^2(M, \partial M; Z),$$

$$\left[\mathrm{S}'\right] \in H_2\left(M, \partial M; Z\right)$$

Moral: The simplicial complex is a data structure that lets us handle everything to do with (co)homology without ever using floating point arithmetic.

Furthermore, when it comes to the Finite Element Method, Whitney forms are the answer to our dreams. ($RW=I$, "$WR \to I$")

Questions:

- Are there other topological gadgets which are readily computed (from these data structures)?
- If so, are the useful?
- Are there topological questions which are easy to ask but are either
  - Undecidable
  - Decidable but not in polynomial time

Answers: To all of the above: Yes!

- To get a better feel for the situation we have to "understand 3-D"

What is Algebraic Topology?

Categories

Categories are “Objects” and “Morphisms” which preserve structure.

Topological Examples:

- Top: Topological manifolds and continuous maps
- PL: Piecewise – linear manifolds and p-l maps
- Diff: Differentiable manifolds and differentiable maps (diffeomorphisms)

Algebraic Examples:

- Group: Groups and group homomorphisms
- Vect: F.D. Vector spaces and linear transformations.

Functors

Maps between categories sending objects to objects and morphisms to morphisms.

Functors preserve the structure of the categories.

Algebraic Topology

Characterization and application of functors from topological to algebraic categories.

i.e. The reduction of topological problems to algebraic problems.

First Favorite Example : (Co)Homology

Prototype Situation: Fundamental Theorem of Calculus

$$\int_{\partial\Omega}\omega = \int_{\Omega} d\omega$$

$$\partial(\partial(\omega)) = 0$$

∂ decreases degree

$$\Downarrow$$

$$\text{Image}(\partial) \subseteq \text{Kernel}(\partial)$$

$$\Downarrow$$

$$\text{H}_\text{p} = \left.\left(\frac{\text{Kernel}(\partial)}{\text{Image}(\partial)}\right)\right|_{\text{p chains}}$$

p^{th} homology group

$$d(d(\omega)) = 0$$

d increases degree

$$\text{e.g.}\begin{cases}\text{div}\,(\text{curl}) = 0\\ \text{curl}\,(\text{grad}) = 0\end{cases}$$

$$\Downarrow$$

$$\text{Image}(d) \subseteq \text{Kernel}(d)$$

$$\Downarrow$$

$$\text{H}^\text{p} = \left.\left(\frac{\text{Kernel}(d)}{\text{Image}(d)}\right)\right|_{\text{p-forms}}$$

p^{th} cohomology group

Key Result:

For coefficients in a field,

$$H_p \simeq H^p$$

and everything reduced to linear algebra

Moral:

(Co)homology is useful and computable – even in n-dimensions! The definitions are natural but not the most intuitive

Second Favorite Example: Homotopy

Prototype Problem: Given a space *X*, with distinguished point x_0, is X homotopic to a point?

Consider mapping p-dimensional spheres into *X*, (with some distinguished point mapped to x_0). Such a mapping form a group called the pth homotopy group of *X*, and is devoted by $\pi_p(X)$. We also call $\pi_1(X)$ the fundamental group.

Fact:
- π_k, k>1 is always abelian
- π_1 can end up being any nonabelian group

One Basic Result: For “reasonable” spaces (e.g. CW complexes), *X* is contractible if and only if $\pi_{\mathrm{k}}(X) = \text{identity, for all k} \geq 1.$

An Evolving Theme: Homotopy groups are related to “simple intuitive questions” but, because π_1 is nonabelian, they may be impossible to compute and basic questions may be undecidable.

Eilenberg – MacLane Spaces

Fact: Given a positive integers n, and a group G, (abelian if n>1), there exists a Space $K(G$,n), unique up to homotopy, such that:

$$\pi_i(K(G_,n)) = \begin{cases} G \;\; if\, i = n \\ O \;\; (Id)\; if\, i \neq n \end{cases}$$

Note: In general, Eilenberg – MacLane spaces cannot have the homotopy type of a manifold. Manifolds with only one nontrivial homotopy group are called aspherical manifolds.

Examples

a) $\mathrm{K}(Z,1) = {R}/{Z}$ $\left(\text{or } S^1 = {R}/{2\pi Z}, \text{ the unit circle}\right)$

b) $\mathrm{K}\left(\pi_1\left(\Sigma g\right),1\right) = \Sigma g$ Where Σg is a compact, orientable, 2-D manifold of genus $g \geq 1$.

c) $\mathrm{K}\left(\pi_1\left(S^3 - k\right),1\right) = S^3 - k$ Where $S^3 - k$ is and knot complement (but not any link complement)

d) $\mathrm{K}(Z,2) = CP^\infty$ i.e. ${S^\infty}/{S^1}$, $\mathrm{K}(Z_2,1) = RP^\infty = {S^\infty}/{Z_2}$

i.e. Projective spaces in Separable Hilbert Spaces over $\begin{Bmatrix} C \\ R \end{Bmatrix}$

Basic Result: Let G be abelian, For “any” Space M, $H^{\ell}(M;G)=[M, K(G,\ell)]$,
Where $[M,N] = \pi_0(Map(M,N))$ is the space of all maps from M to N, up to homotopy.

This can be useful: $H^1(M,Z)=[M,S^1]$ given that $K(Z,1)=S^1$ where M is any orientable compact, n-dimensional manifold with boundary. Furthermore, for a map $f : M \to S^1$ representing a particular cohomology class,

$$\left[f^*\left(\frac{d\theta}{2\pi}\right)\right] \in H^1(M;Z)$$

and any regular value p □ S^1, $f^{-1}(p)$ is an orientable embedded manifold , with boundary, whose relative homology class

$$\left[f^{-1}(p)\right] \in H_{n-1}(M,\partial M;Z)$$

is the Lefschetz dual to $\left[f^*\left(\frac{d\theta}{2\pi}\right)\right]$

or, more concretely

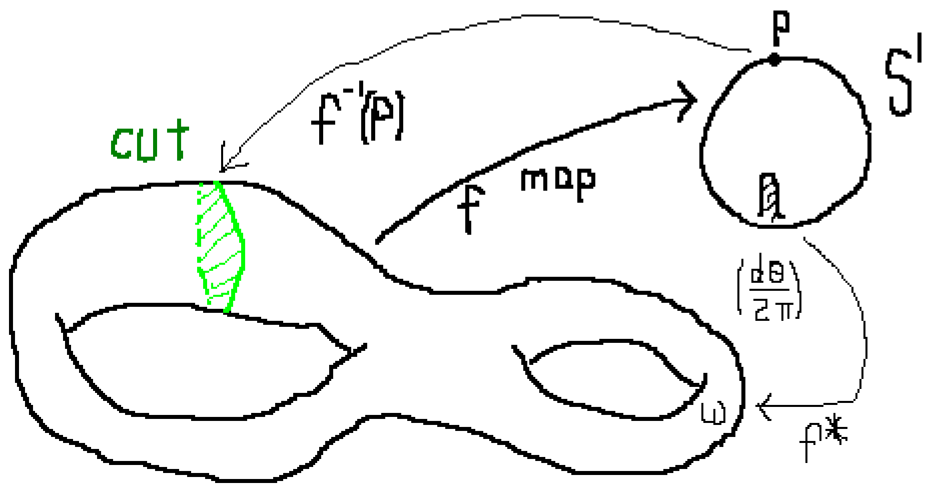
If ω is a closed form with integer periods,

$$f(p) = e^{2\pi i \int_{P_0}^{P} \omega} : M \to S^1$$

$$f^*\left(\frac{d\theta}{2\pi}\right) = \frac{1}{2\pi i} d(\ln f) = \omega$$

"cuts" are just level sets of f^{-1}:

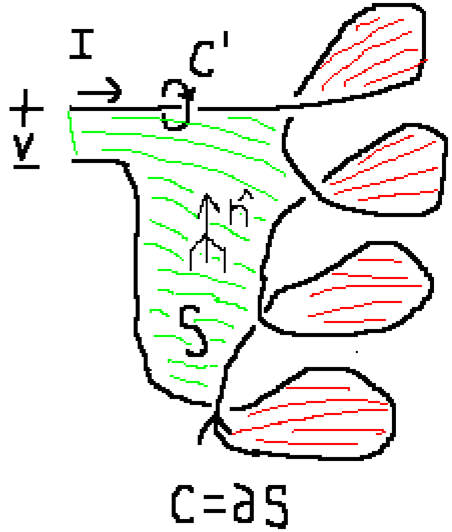
In 3-D



Inductance Calculations and Potentials

$$L = \frac{\Phi}{\mathrm{I}}$$

$$\mathrm{I} = \int_{c'} H \cdot d\lambda$$



R = Complement of wire relative to R^3

$$\Phi = \int_s B \cdot \hat{n} dS = \int_{c=\partial s} A \cdot d\lambda$$

$$[\mathrm{c}'] \neq 0 \text{ in } \mathrm{H}_1(\mathrm{R}, \mathrm{Z}),$$

$$[\mathrm{S}] \neq 0 \text{ in } \mathrm{H}_2(\mathrm{R}, \partial\mathrm{R}; \mathrm{Z})$$

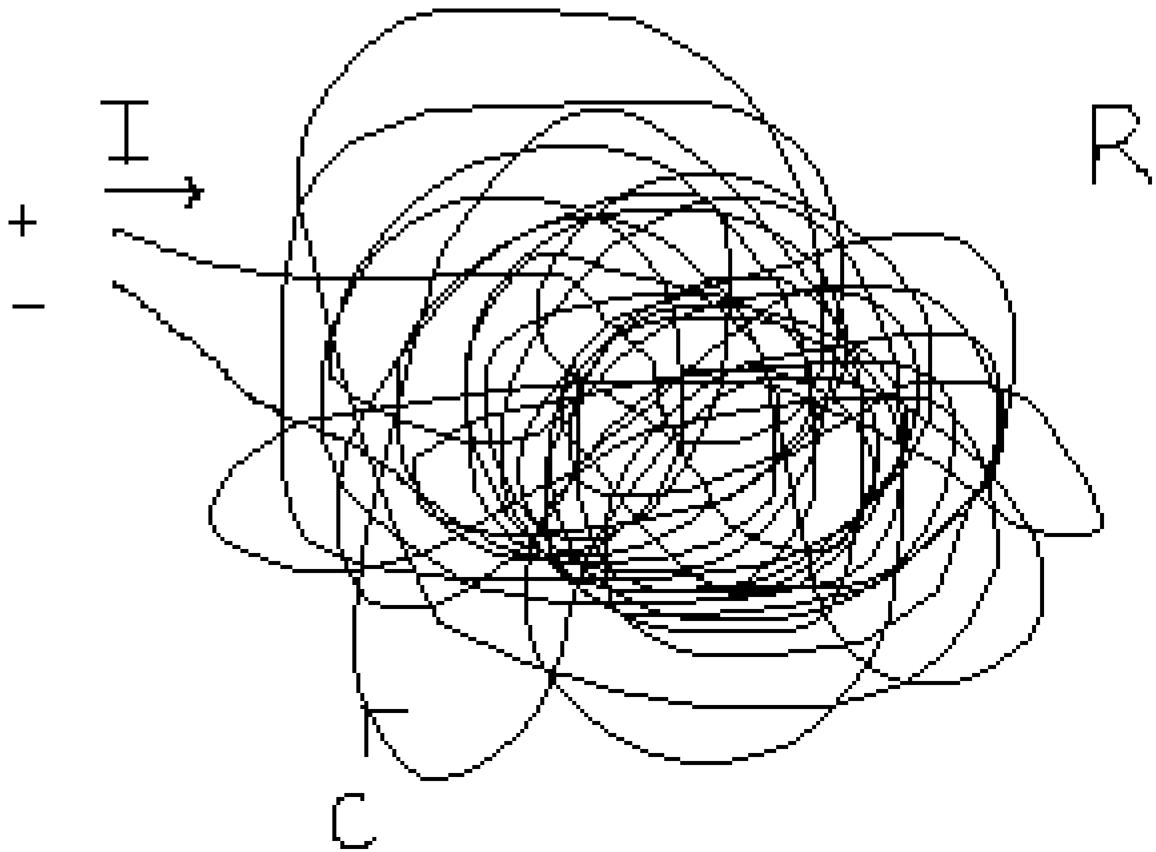
To compute inductance, we use a vector or a scalar potential.

Inductance Calculations and Potentials

Moral: If we use a scalar potential, we need the cut for two reasons:

- To define a single valued scalar potential
- To find the flux if we use first principles (as opposed to an energy functional)

Consider a “current carrying ball of wool” or some complicated current carrying surface.



Facts:

1. $C=\partial S$ where S is an orientable embedded 2-D manifold with boundary
2. There is an algorithm to compute S as a level set of a harmonic map ($R \rightarrow S^1$). It has:
   - Time complexity:

     $O(n^2)$+ Complexity of "solving" an elliptic p.d.e.
   - Space complexity:

     $O(n^{4/3})$ # nodes in FE mesh required for a solution to the harmonic map problem.
3. It appears that no exact arithmetic algorithm exists which runs in polynomial time! (belted trees are not embedded manifolds.

Notes: H_* is all we need, but optimality (reordering) requires more sophisticated topological information.

π^* vs. H^* Algebraically

- π_i is abelian for $i>1$, and for $i\geq 1$ we have a Hurewicz map : $h_i : \pi_i \to H_i$

- $i=1$; $h_1: \pi_1 \to H_1$ is onto, kernel $h_1=[\pi_1, \pi_1]$
  $\to H_1=\pi_1/[\pi_1, \pi_1]$ (Poincaré)

- $i=2$; $h_2: \pi_2 \to H_2$ (typically neither 1-1 nor onto)
  - Ker h_2 – spheres which bound 3-manifolds in R which are not balls.

π* vs. H* Algebraically

- H_2/Image(h_2) –homology classes in R necessarily having handles when realized by orientable embedded manifolds.

- Thurston Norm on (H_2/Image(h_2)) □ R (uniquely 3-D)

Computation and the Thurston Norm

In general the Thurston norm requires exponential time to compute. However "tight" bonds are computable in polynomial time. This is key for trying to make the "cuts on handles of cuts" idea (Haken Hiearchy) work.

Lower Central Series (l.c.s.(π_1))

- Recall $H_1=\pi_1/[\pi_1, \pi_1]$
- Define:

$$\pi_1^1 = \pi_1$$
$$\pi_1^2 = \left[\pi_1^1, \pi_1^1\right] = \left[\pi_1, \pi_1\right]$$
$$\pi_1^3 = \left[\pi_1^1, \pi_1^2\right] = \left[\pi_1 \left[\pi_1, \pi_1\right]\right]$$

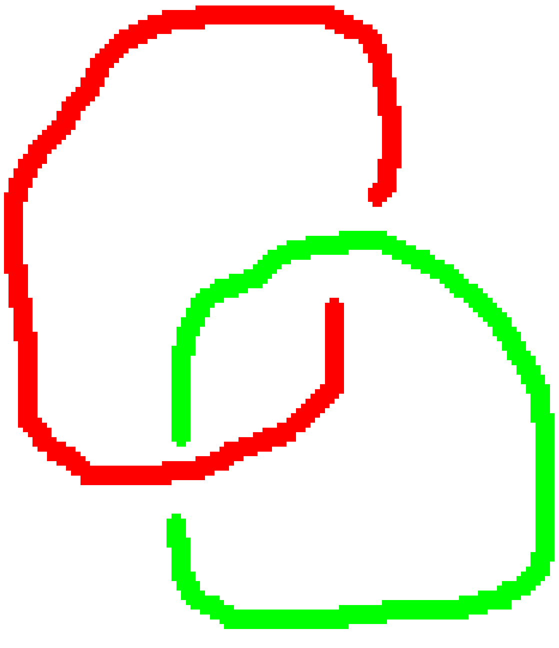
M

$$\pi_1^k = \left[\pi_1^1, \pi_1^{(k-1)}\right] \qquad \Rightarrow G_k = \pi_1^k \big/ \pi_1^{(k-1)} \; Abelian \left(G_1 = H_1\right)$$

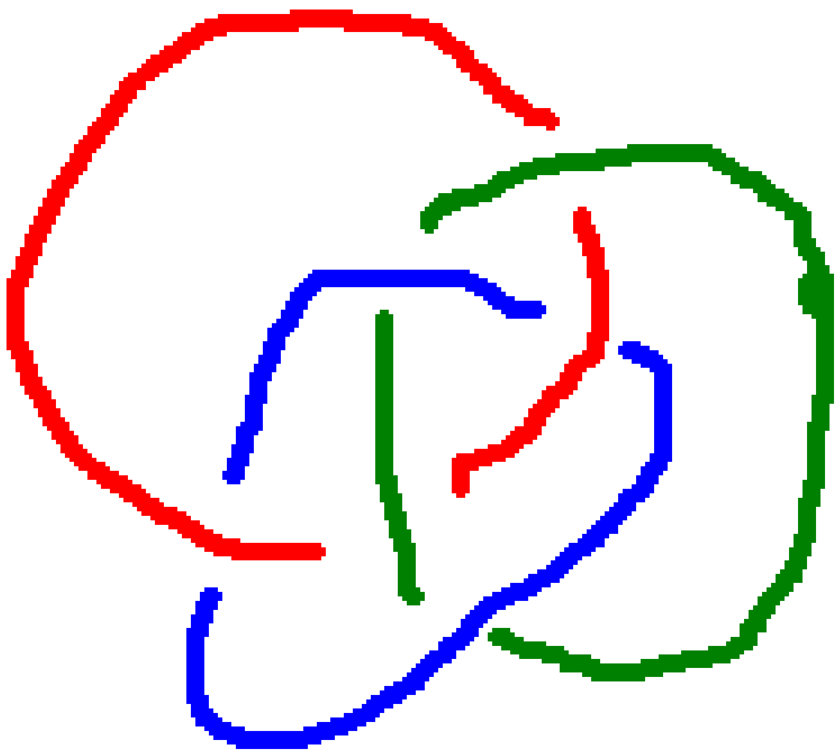
LCS, Continued

- This “data” is equivalent to (in 3-D):
  1. Massey products in the cohomology ring
  2. Differential graded Lie Algebras found in “the minimal models of rational homotopy theory”
  3. Chen’s iterated integrals for computing loop space homology.
- The l.c.s. is computable in polynomial time from FE data structures (to a given depth). Note that π_1 doesn’t seem to have this property
- Remark: l.c.s(π_1) is very effective in detection tangling, which in turn is related to π_2 being trivial.

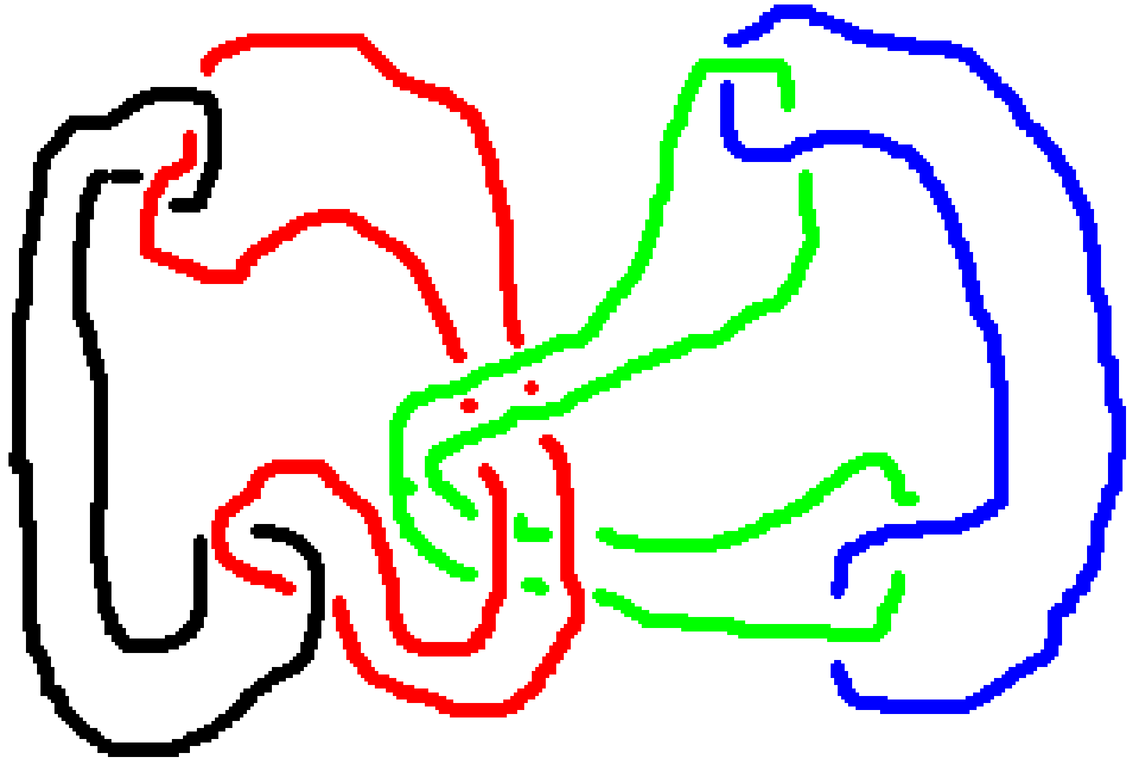
Detected at the second level:



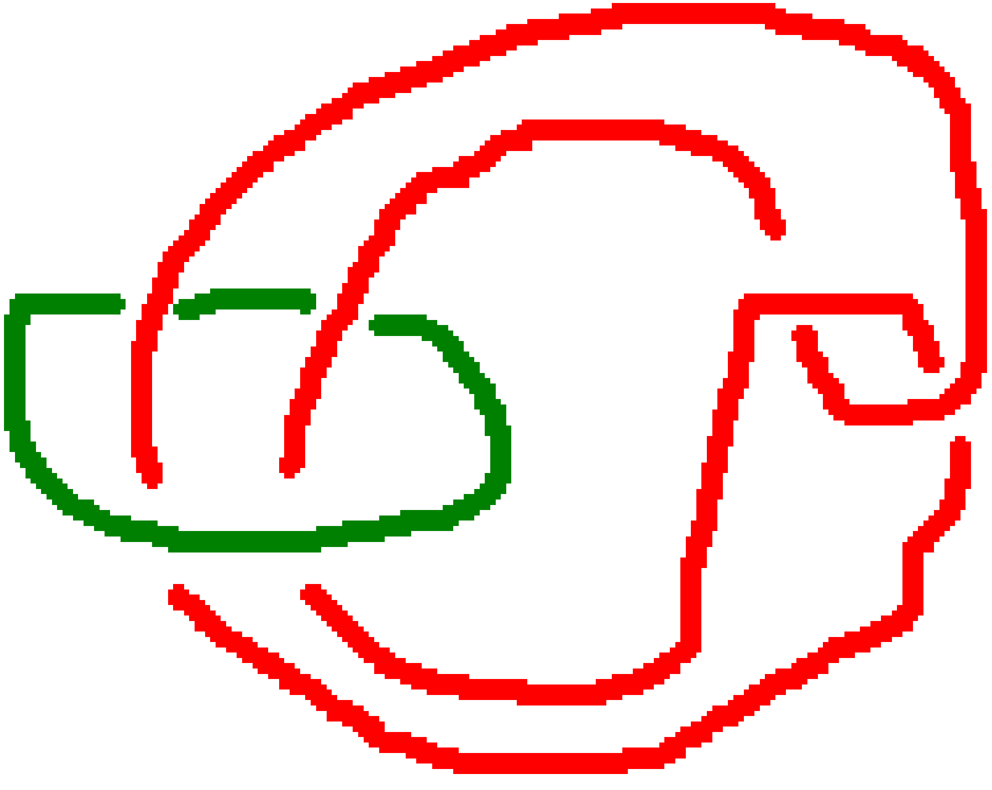
Detected at the third level:



Detected at the fourth level:



Detected at the fourth level:



…Can always find n+1 tangled curves which become untangled if any one is removed. This is detected at the n+1st level.

Fact: l.c.s(π_1) being trivial is necessary but not sufficient for π_2 to be trivial.

Tangled (π_2 trivial) but tangling cannot be detected at any level of the l.c.s(π_1)

Derived (normal) Central Series (again $H_1=\pi_1/[\pi_1, \pi_1]$)

$$\pi_1^{(0)} = \pi_1$$

$$\pi_1^{(1)} = \left[\pi_1^{(0)}, \pi_1^{(0)}\right] = \left[\pi_1, \pi_1\right]$$

$$\pi_1^{(2)} = \left[\pi_1^{(1)}, \pi_1^{(1)}\right] = \left[\left[\pi_1, \pi_1\right], \left[\pi_1, \pi_1\right]\right]$$

$$\pi_1^{(3)} = \left[\pi_1^{(2)}, \pi_1^{(2)}\right] = \mathrm{K}$$

M

$$\textit{Let} \quad \mathrm{G}_{\mathrm{k}} = \frac{\pi_1^{(k-1)}}{\left[\pi_1^{(k-1)}, \pi_1^{(k-1)}\right]} \quad \left(\mathrm{G}^{(1)} = H_1\right)$$

Related to homology groups of iterated maximal abelian covering spaces.

- This articulated the “cuts upon handles of cuts picture. Although it is very geometric, it is however, not computable in polynomial time.
- Moral: (and punch line) We need to effectively approximate this picture in polynomial time if we are able to get a CAD system to communicate topological aspects of knotted geometries to a user.

Dimensionality and Issues of Computability and Decidability

Sample Questions:

- Can we classify n-dimensional manifolds up to homeomorphism? (i.e. make a list)
- If so, is there an algorithm which takes a given manifold and finds its homeomorphism type on the list?

Simplest Case:

I-D

Theorem: Every compact connected one-dimensional manifold is homeomorphic to the circle.

i.e. the list has one entry

Also,

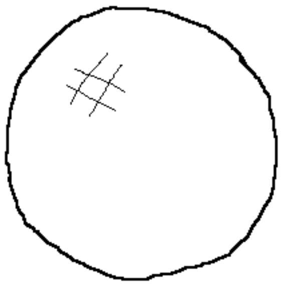
$$\begin{cases} \pi_1(S^1) \simeq Z & \text{(winding number)} \\ \pi_k(S^1) = Id & k>1 \end{cases}$$

(R is a covering space and contractible)

2-D There is a classification of compact, connected, orientable surfaces, up to homeomorphism, and an algorithm to identify the homeomorphism type.

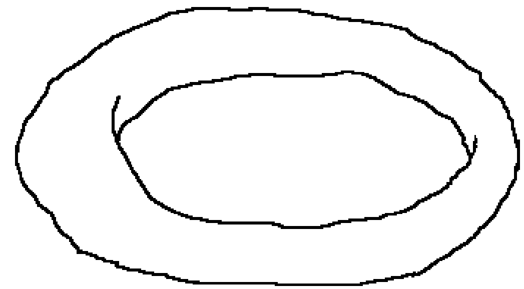
List:

Σ_0



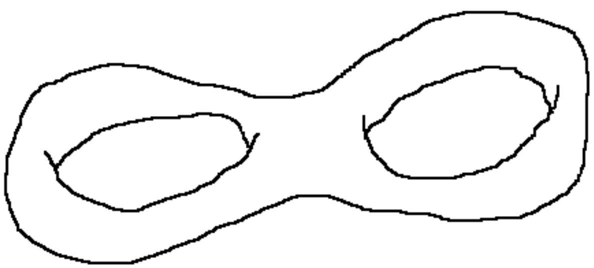
$g = 0$

Σ_1



$g = 1$

Σ_2



$g = 2$

Σ_3

$g = 3$

… and so on.

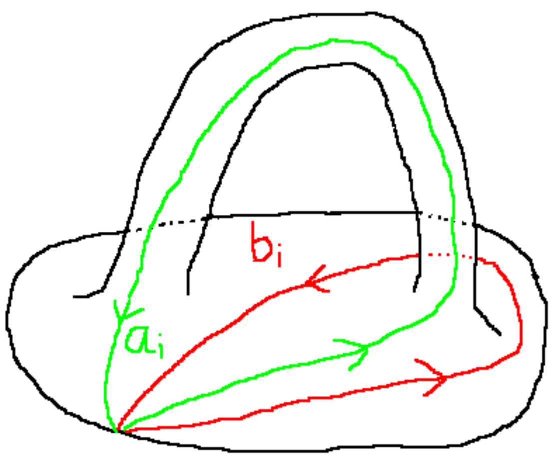
How to find a surface on the list

Given S

- Triangulate
- Let $\alpha_i = \# i - \text{simplices}, 0 \leq i \leq 2$
- Compute $\chi(S) = \sum_{i=0}^{2} (-1)^i \alpha_i$
- $2p = 2 - \chi(S)$
- Conclude $S = \Sigma g$
- Find Σg in the list (count)

<u>Conclusion</u>: In 2-D the problem is decidable and computable

<u>Also</u>:
χ(S) determines *S* up to diffeomorphism



$$\begin{pmatrix} \pi_1(\Sigma_g)\text{ generated by } \{\alpha_i, \beta_i\}_{i=1}^{g}, \\ \text{and } \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \text{ is the only relation} \\ \pi_k(\Sigma_g) = id.\text{ for k} \geq 2 \end{pmatrix}$$

$\Rightarrow \chi(S)$ determines homotopy type

4-D

Observe: $\partial B^k = S^{k-1}$,
and $$\partial(B^2 \times S^2) = S^1 \times S^2 = \pm\partial(S^1 \times B^3)$$

Thickened sphere in 4-D (for $B^2 \times S^2$) — Tubular neighborhood. of S^1 in 4-D (for $S^1 \times B^3$)

Cool Fact: (Seifert-Threlfall (see Massey)) Given a finite presented group G, there is a compact 4-manifold M^4 such that $\pi_1(M^4) = G$

Problem: Given a compact 4-manifold N^4, is there an algorithm that would identify N^4 (up to homeomorphism) on list of all possible compact 4-manifolds?

Cool Fact:

Any such algorithm would solve the word problem for finitely presented groups.

But:

The word problem for finitely presented groups is equivalent to the halting problem for a Turing machine.

And:

The halting problem in undecidable, so…..

Theorem/Moral/Conclusion: (Markov)

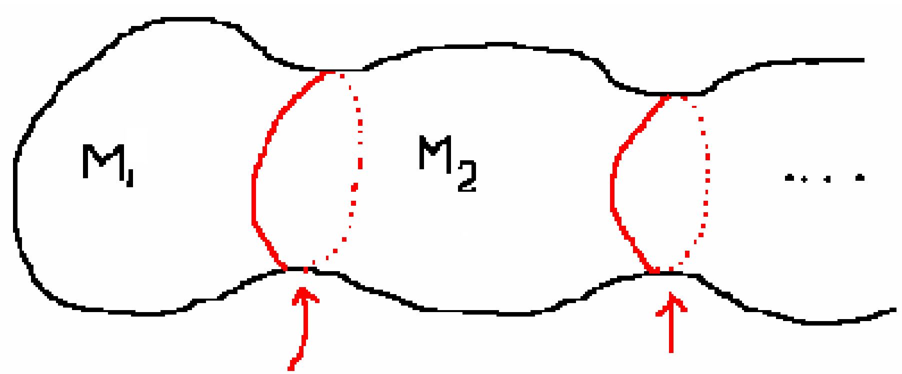
There does not exist an algorithm which solves the classification problem in 4-D.

That is, the problem is undecidable and uncomputable.

3-D (Modulo the Poincaré Conjecture in dimension 3), There is a classification of (compact connected, oriented) 3-manifolds and a set of invariants which distinguish
homotopy/ homeomorphism/ diffeomorophism types.

Knesser 1929, $\partial M = \square$ Milnor 1962, $\partial M \neq \square$ Hempel $\partial M \neq \square$

M^3 looks like: $M_1 \# M_2 \# M_3 \# \ldots\ldots \# M_k$ "M_i is $K(\pi_1(M_i), 1)$"



$S^2 \in \pi_2(M^3)$ $S^2 \in \pi_2(M^3)$

Invariants:

determine homotopy type:

1) $\pi_1(M^3)$
2) Image [M] under $\mu : H_3(M^3) \to H_3(\pi_1(M^3))$
3) Torsion Invariant

(all three) determine homeomorphism type

Problem: Given an N^3, can we produce a list of all possibilities (Moise: it exists!) and, (assuming Poincaré conjecture), can we find an algorithm that finds N^3 on the list? (should be possible)

Decidable. All invariants are computable and presentable in terms of FE data structures but, not easily computable, because of the fundamental group.

Intermediate Lessons Learned:

(Co)homology: Easy in n-dimensions, “just linear algebra.”
“Kirchhoff is happy”

π_1(M) in 3-D: Decidable, but in general, there are questions not resolvable in polynomial time.

l.c.s π_1(M) in 3-D: *i.e.*

$$G_k = \frac{\pi^{(k)}}{[\pi_1, \pi^{(k)}]}, \textit{inductively}, \ \pi^{(0)} = \pi_1, \ G_0 = H_1 (\textit{Poincaré})$$

- Of interest in MHD.
- Computable in polynomial time (via Massey Products in H^*).

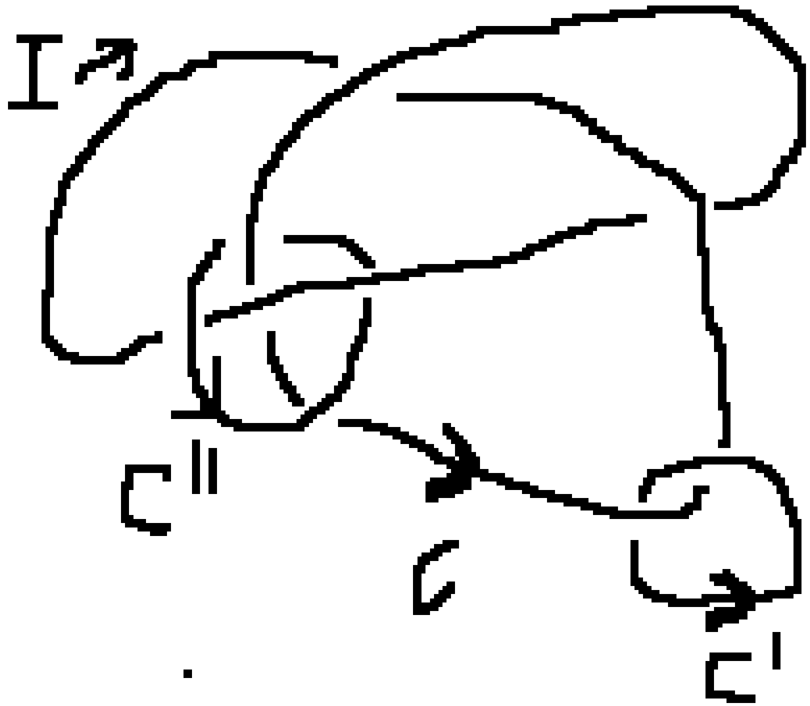
Alexander Ideal: (*i.e.* π_1 acting on $Z(\pi_1)$ – group ring.)

- Computable
- Of interest for force-free fields.

Reidemeister Torsion: Not a topological invariant if $\partial M \neq$ □, but related to the determinant of the D-N map of Impedance Tomography (sensitivity analysis).

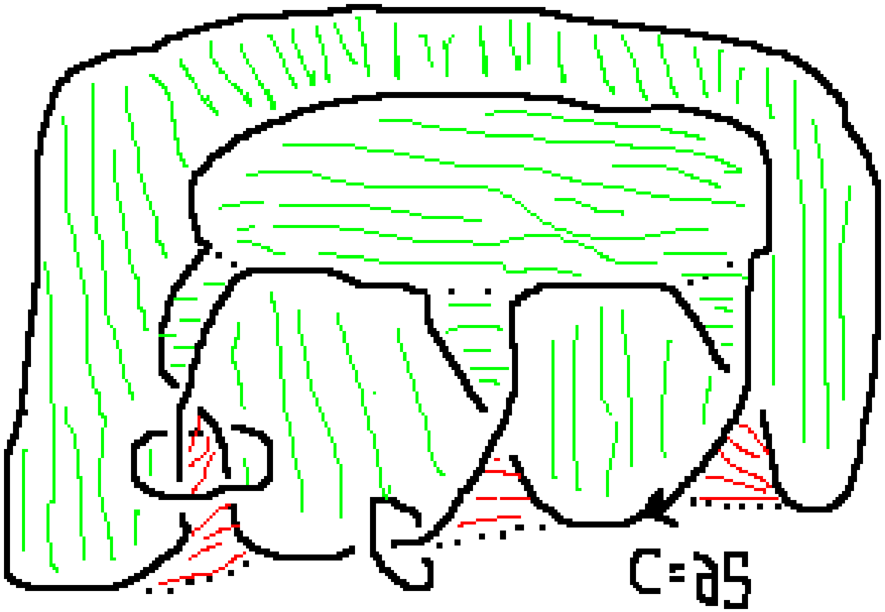
Cuts for magnetic scalar potentials: Computable via real arithmetic and FEA of harmonic maps into S^1. Certain criteria for “simplest cuts” not be verifiable in polynomial time.

π_1 vs. H_1 Heuristically – Consider the simplest Knot:



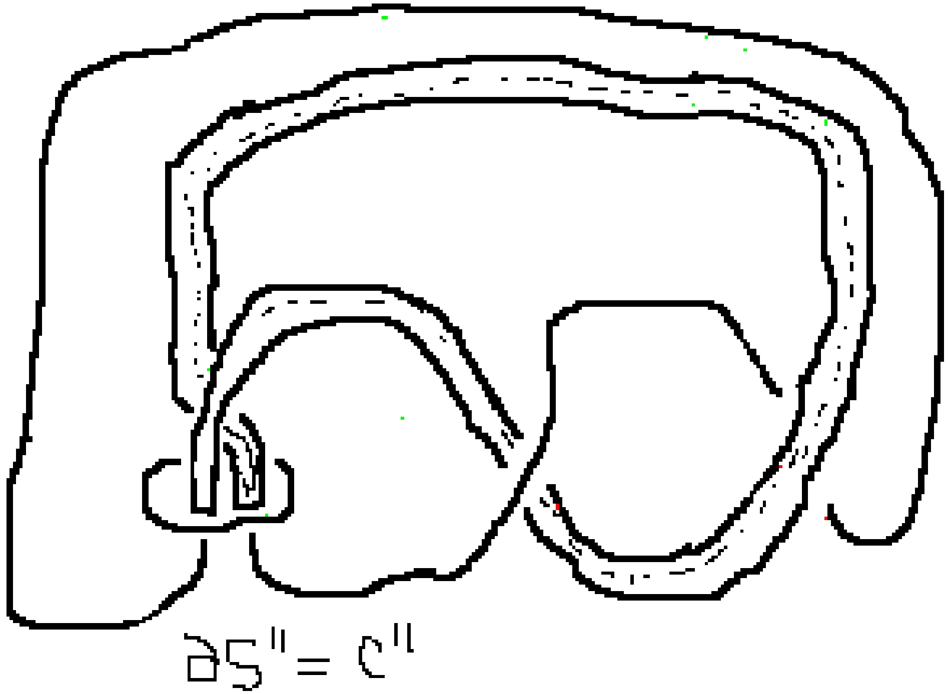
$$c' : \oint_{c'} H \cdot d\lambda = \mathrm{I}$$

Consider the simplest knot as a boundary of a surface



$$c' : \oint_{c'} H \cdot d\lambda = \mathrm{I}$$

C” is interesting….



$$c'': \oint_{c''=\partial S''} \mathrm{H} \cdot \mathrm{d}\lambda = \int_{S''} curlH \cdot \hat{n} dS = 0$$

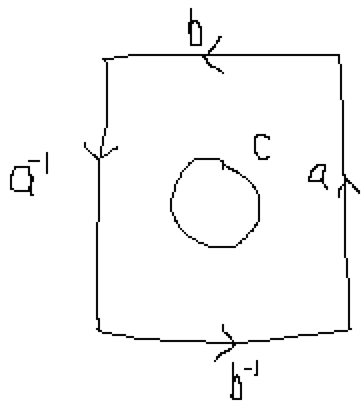
S'' lies in the region where curl$H = 0$

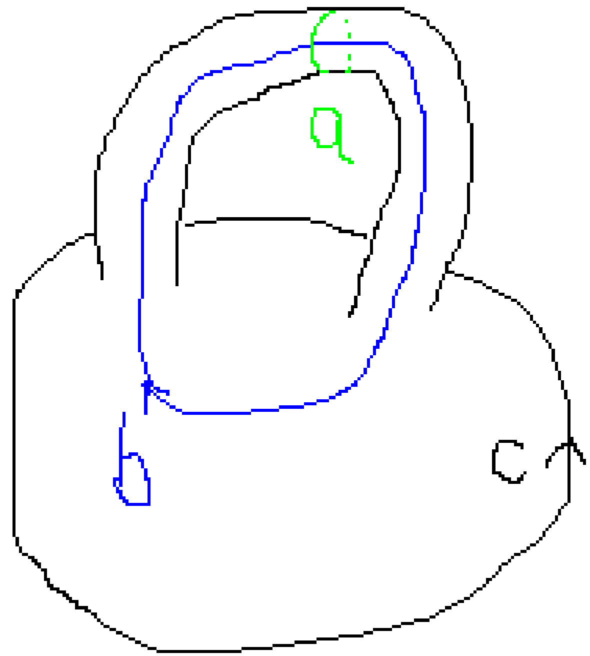
Why C” is interesting…

- Let π_1 be the group generated by “based loops,” the product is given by composition of loops and there is the homotopy equivalence relation.(~)
- Boundaries of discs are trivial in π_1.
- Boundaries of “discs with handles are commutators in π_1.

$$c \sim aba^{-1}b^{-1}$$

$$c \sim [a,b]$$





1) $[\pi_1, \pi_1]$ is trivial in homology but not necessarily in homotopy
2) Poincaré said: $H_1 = \pi_1/[\pi_1, \pi_1]$

Question: How do we think about $[\pi_1, \pi_1]$?

Inspired by "Abelian covering spaces," we can let the cuts be current carrying sheets and consider:

$$\pi_1\left(\widetilde{R}\right) \text{ where } \widetilde{\mathrm{R}} = \mathrm{R} - \bigcup_{\mathrm{i}=1}^{\beta_1(R)} S_i$$

← Betti number

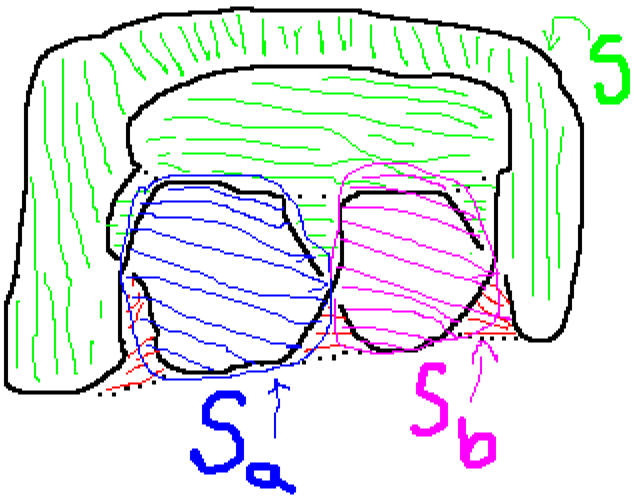
Assume cuts are made so as to minimize $\beta_1\left(\widetilde{R};Z\right)$

Poincaré says: $H_1\left(\widetilde{R}\right)=\pi\left(\widetilde{R}\right)/\left[\pi_1\left(\widetilde{R}\right),\pi_1\left(\widetilde{R}\right)\right]$

Question: How does one measure or comprehend the complexity of π_1(R)?

Example - For the trefoil:

π_1(R-S) = [π_1(R)$_1\pi_1$(R)]
β_1(R-S) =2



The cuts S_a and S_b for H_1(R-S;Z) are discs and π_1(R-(SUS_aUS_b)) is trivial.

If S_a or S_b had to have handles we could iterate the construction one more time

i.e. make cuts on $$\tilde{\tilde{R}} = \tilde{R} - S_a \cup S_b = R - (S \cup S_a \cup S_b)$$

Question: If we could find minimal genus cuts, then does iteration the construction of making cuts on the handles of cuts eventually make the remaining space simply connected?

Answer: No – in general, but Yes – if π_2 is trivial. (R is “Haken” in this case)

Problems: There is no polynomial time algorithm to:

1) Check if π_2 is trivial
2) Check to see if a cut has minimal genus.

<u>and Hopes:</u>

1. Obstructions to π_2 being trivial can be found in l.c.s(π_1) - a computable measure of $[\pi_1, \pi_1]$ complexity.
2. An “Alexander Norm” can be computed in polynomial time to get a tight bound on the “Thurston Norm” (which counts the genus of the cuts.) (McMullen)
3. Recent work of Shelly Harvey refines this work.