HOMOTOPICAL PROPERTIES FOR CONCURRENT SYSTEMS

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HOMOTOPICAL PROPERTIES FOR CONCURRENT SYSTEMS

FROM A STRUCTURAL APPROACH ...

A ROADMAP ...

OUTLINE

I Motivations and Background on the Homology of Rewriting

II Rewriting Logic and Crossed Modules

III Identities Among Relations

IV Works in progress

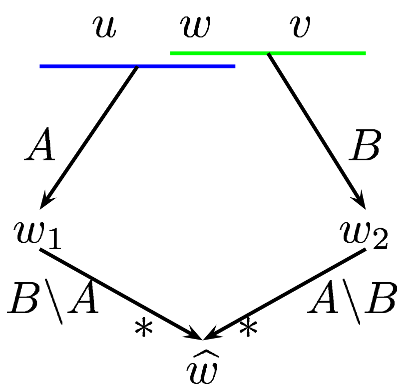
I. MOTIVATIONS

From Rewriting to Homology

- A string rewriting system $< \Sigma \mid R >$ is given by
  - An alphabet $\Sigma = \{a, b, c, \dots\}$
  - A set of rules $R = \{A, B, C, \dots\} \subset \Sigma^* \times \Sigma^*$, $\quad u \xrightarrow{A} v$

- Conflicts in a confluent $< \Sigma \mid R >$ correspond to the following situations :

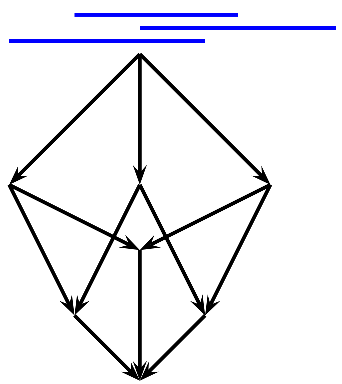
Critical pair :



$uw \xrightarrow{A} w_1$

$wv \xrightarrow{B} w_2$

Critical triple :



$\cdots$ n-overlapping ambiguities $\cdots$

- Squier's resolution, [Squier'87], $\mathbf{M} \simeq \Sigma^*/R$:

$$\mathbb{Z}\mathbf{M}[\text{Critical pairs}] \longrightarrow \mathbb{Z}\mathbf{M}[R] \longrightarrow \mathbb{Z}\mathbf{M}[\Sigma] \longrightarrow \mathbb{Z}\mathbf{M} \longrightarrow \mathbb{Z} \qquad \in \mathsf{Ab}^{\mathbf{M}}$$

$$\langle A,B\rangle \longmapsto A + B\backslash A - A\backslash B - B$$

1. Fox Differential Calculus on terms

▶ Terms

▶ A term rewriting system (TRS) is a pair $< \Omega \mid R >$, where

- $\Omega = (\Omega(\mathbf{n},\mathbf{1}))_{n\in\mathbb{N}}$ is an equational signature,
- R is a set of rules $u \to v$, where $u,v \in T_\Omega(X)$, the free Ω-algebra given inductively :

$$\frac{}{x \in T_\Omega(X)}\ x \in X, \qquad \frac{t_1, \dots, t_n \in T_\Omega(X)}{f(t_1, \dots, t_n) \in T_\Omega(X)}\ f \in \Omega(\mathbf{n},\mathbf{1})$$

such that, $u \notin X$, $\mathcal{V}ar(v) \subseteq \mathcal{V}ar(u)$.

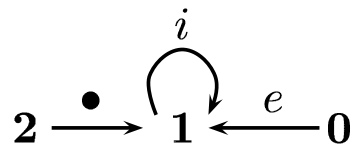
Example : $\Omega = \{\ \mathbf{2} \xrightarrow{\bullet} \mathbf{1}\ \}$, $R = \{\ (x \bullet y) \bullet z \to x \bullet (y \bullet z)\ \}$.

▶ An algebraic theory is a small category $\mathbb{A}$, whose set of objects consists of countably many objects $\mathbf{0}, \mathbf{1}, \dots, \mathbf{n}, \dots$ and in which each object $\mathbf{n}$ is the product of the object $\mathbf{1}$ with itself n times.

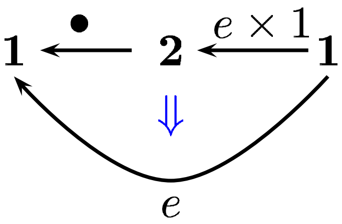
Proposition. (**Abstract Syntax**) Any TRS $< \Omega \mid R >$ has a complete simulation on the free algebraic theory $\mathbb{F}(\Omega)$.

▶ **Example.** The theory of groups

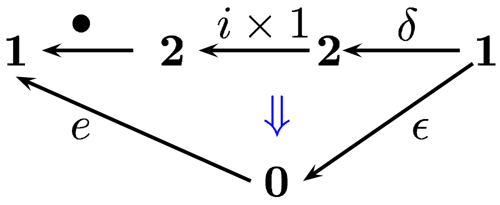
$\Omega =$



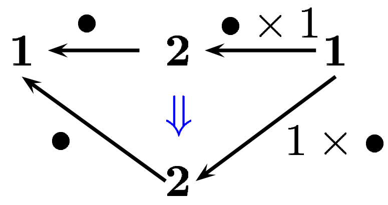
$e \bullet x \to x$



$i(x) \bullet x \to e$



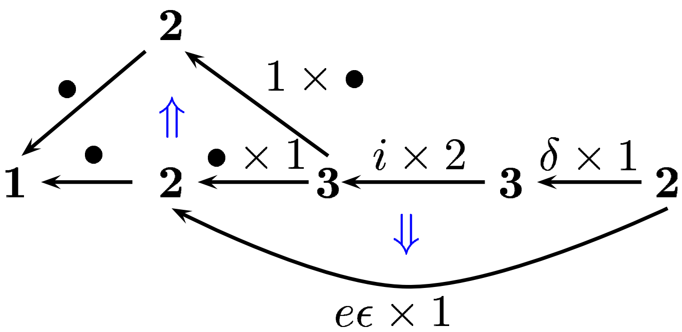
$(x \bullet y) \bullet z \to x \bullet (y \bullet z)$



The critical pair

$$z \longleftarrow e \bullet y \longleftarrow (i(x) \bullet x) \bullet y \longrightarrow i(x) \bullet (x \bullet y)$$

corresponds to the following critical pair on $\mathbb{F}(\Omega)$:



▶ Cartesian natural systems

▶ Which abelian category for computing a projective resolution for $< \Omega \mid R >$?

$$\mathsf{Ab}(\mathsf{Th}/\mathbb{A}) \supset \mathsf{Ab}^{\mathbb{A}^{op}\times\mathbb{A}} \supset \mathsf{Ab}^{\mathbb{A}}$$

▶ The category of factorisations $F\mathbb{A}$ is the category defined by :

$$\mathbf{Ob}(F\mathbb{A}) = \mathbf{Mor}(\mathbb{A}), \qquad F\mathbb{A}(w,w') \ni \begin{array}{ccc} \mathbf{n'} & \xrightarrow{u} & \mathbf{m'} \\ w \uparrow & & \uparrow w' \\ \mathbf{n} & \xleftarrow[v]{} & \mathbf{m} \end{array}$$

A cartesian natural system on $\mathbb{A}$ is a functor $D : F\mathbb{A} \longrightarrow \mathsf{Ab}$, preserving finite products :

$$D_w \simeq D_{\pi_1 w} \times D_{\pi_2 w}, \quad \text{for all} \quad \mathbf{m} \xrightarrow{w} \mathbf{n_1} \times \mathbf{n_2} \xrightarrow{\pi_i} \mathbf{n_i}$$

Denote $\mathsf{Ab}^{F\mathbb{A}}$ the category of such functors and natural transformations.

- The category $\mathsf{Ab}^{F\mathbb{A}}$ is abelian with enough projectives.
- The free cartesian natural system generated by a subset X of morphisms in $\mathbb{A}$ is :

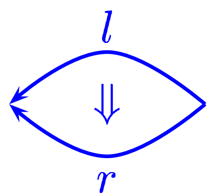
$$\mathbb{Z}\mathbb{A}[X]_- = \bigoplus_{x\in X} \mathbb{Z}F\mathbb{A}(x,-).$$

▶ From a complete (noetherian and confluent) TRS $< \Omega \mid R >$, we construct a free acyclic resolution of $\mathbb{Z}$ in the category $\mathsf{Ab}^{F\mathbb{A}}$, where $\mathbb{A} = \mathbb{F}(\Omega)/R$.

▶ FOX DIFFERENTIAL CALCULUS

▶ The natural system H of 2-homological syzygies of $< \Omega \mid R >$ is defined as the kernel of the Fox jacobian :

$$0 \longrightarrow \mathrm{H} \longrightarrow \mathbb{Z}\mathbb{A}[R] \xrightarrow{d} \mathbb{Z}\mathbb{A}[\Omega] \qquad \in \mathsf{Ab}^{F\mathbb{A}}$$



$$\longmapsto \sum_{x \in \Omega} \left(\frac{\partial l}{\partial x} - \frac{\partial r}{\partial x} \right),$$

Theorem. If $< \Omega \mid R >$ is complete, then the natural system H is generated by critical pairs.

In particular, if R is finite, then $\mathrm{H}_3(\mathbb{A}, \mathbb{Z}) \simeq \mathrm{Tor}_3^{F\mathbb{A}}(\mathbb{Z}, \mathbb{Z})$ is finitely generated.

▶ More generally by an inductive process, [Anick '86, Kobayashi '90] :

If $< \Omega \mid R >$ is a complete TRS, there is an acyclic free resolution of natural systems

$$\cdots \longrightarrow \mathbb{Z}\mathbb{A}[A_n] \xrightarrow{d_n} \mathbb{Z}\mathbb{A}[A_{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}\mathbb{A}[A_2] \xrightarrow{d_2} \mathbb{Z}\mathbb{A}[R] \xrightarrow{d} \mathbb{Z}\mathbb{A}[\Omega] \xrightarrow{\epsilon} \mathbb{Z} \qquad \in \mathsf{Ab}^{F\mathbb{A}}$$

where A_n is the set of n-overlapping and d_n decomposes n-overlapping into $n-1$-overlapping.

Hence, if R is finite then $\mathrm{H}_n(\mathbb{A}, \mathbb{Z})$ is finitely generated for any n.

▶ **Example.** $< \mathbf{2} \xrightarrow{\bullet} \mathbf{1} \mid (x \bullet y) \bullet z \xrightarrow{A} x \bullet (y \bullet z) >$,

- 2-syzygy $\equiv$ Stasheff-MacLane pentagon,
- higher syzygies $\equiv$ higher Stasheff polytopes.

▶ **AIMS :** Apply Fox calculus to systems such as :

- Concurrent models (ChAM, Petri nets, ...),
- Functional models (λ-calculus, ...),

▶ A prototypical example : the λ-calculus.

The $\lambda\sigma$-calculus as an algebraic setting for the λ-calculus, [Abadi, Cardelli, Curien, Lévy '91].

- $\mathbb{S}$ the algebraic theory of explicit substitution :

`Terms` $a,b ::= 1 \mid ab \mid \lambda a \mid a[s]$

`Substitutions` : $s,t ::= \mathrm{id} \mid \uparrow \mid a.s \mid s \circ t$.

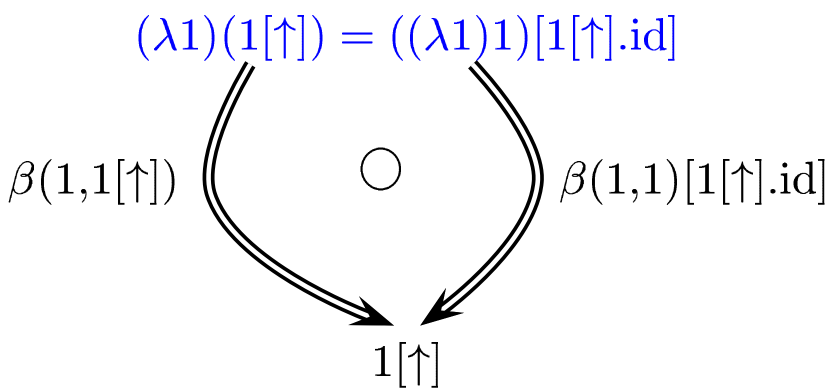
`Axioms` :

$$1[\mathrm{id}] = 1, \quad 1[a.s] = a, \quad (ab)[s] = (a[s])(b[s]), \quad (\lambda a)[s] = \lambda(a[1.(s\circ \uparrow)]), \quad a[s][t] = a[s \circ t],$$

$$\mathrm{id} \circ s = s, \quad \uparrow \circ \mathrm{id} = \uparrow, \quad \uparrow \circ (a.s) = s, \quad (a.s) \circ t = a[t].(s \circ t), \quad (r \circ s) \circ t = r \circ (s \circ t).$$

- β rule : $\beta(u,v) : (\lambda u)v \longrightarrow u[v.id]$.

The simulation of λ-calculus on $(\mathbb{S},R)$ is not complete, [Laneve-Montanari '96] :



2. Syzygies for rewriting logic

- The plan is :

  1. Semantically describe a concurrent or functional model, by a rewriting logic $(\mathbb{S}, R)$ (rewriting modulo),

  2. Associate to $(\mathbb{S}, R)$ the free crossed module of algebraic theories

$$\mathbb{C}(R) \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \mathbb{S}$$

  3. Use the one-to-one correspondence

$$\{\text{critical pairs}\} \approx \{\text{identities among relations}\} \approx \{\text{2-homological syzygies}\}$$

  4. Compute identities among relations and higher order syzygies.

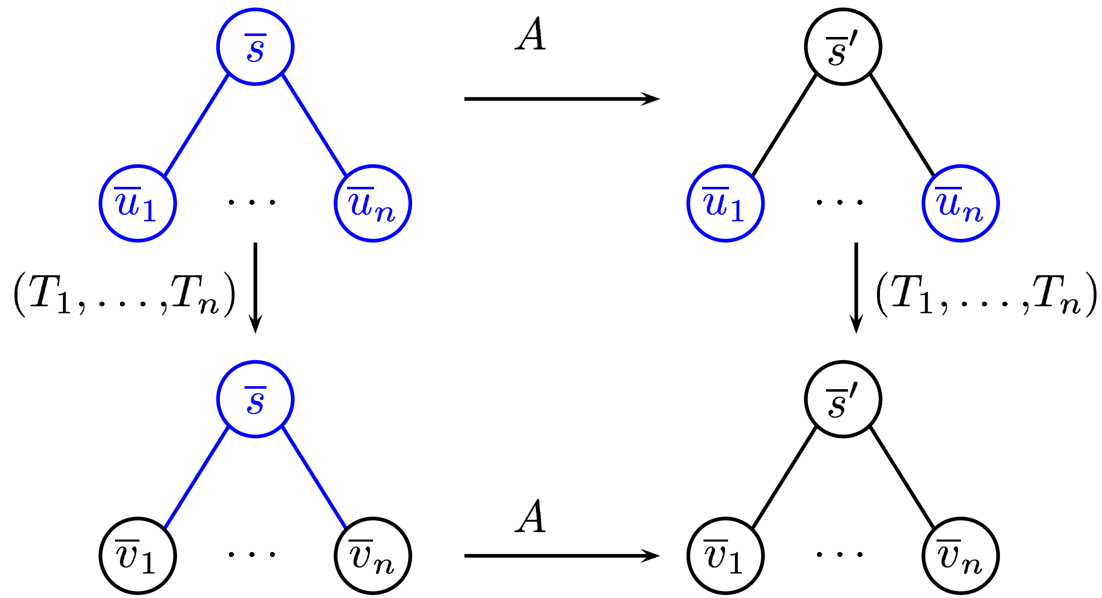
II. Rewriting logic and crossed modules

1. Rewriting Logic

Model of concurrent system	$\rightsquigarrow$	Rewrite Theory,
States with structure (multiset, tree , . . .)	$\rightsquigarrow$	Terms in an algebraic theory $\mathbb{S}$,
Atomic local transition	$\rightsquigarrow$	Rewriting rule on $\mathbb{S}$,
Transition : $s_0 \longrightarrow s_1$	$\rightsquigarrow$	Rewriting : $\overline{s}_0 \longrightarrow \overline{s}_1$.

$$\overline{s} \xrightarrow{A} \overline{s}', \qquad \overline{u}_i \xrightarrow{T_i} \overline{v}_i$$

The simultaneous tansition on $\overline{s}(\overline{u}_1, \cdots, \overline{u}_n)$ is represented by



▶ **Definition.** [Meseguer '90] A rewrite theory $(\mathbb{S}, R)$ is given by:

- an order-sorted algebraic theory $\mathbb{S}$, (finitely) presented by $< \Omega \mid E >$,
- a set of rules R on $\mathbb{S}$, modulo E.

Definition. The rewriting relation $\longrightarrow$ on $\mathbb{S}$ is defined inductively :

- Reflexivity, transitivity:

$$\frac{}{\overline{s} \longrightarrow \overline{s}}\ \overline{s} \in \mathbb{S} \qquad \frac{\overline{s} \longrightarrow \overline{s}' \quad \overline{s}' \longrightarrow \overline{s}''}{\overline{s} \longrightarrow \overline{s}''}\ \overline{s},\overline{s}',\overline{s}'' \in \mathbb{S}$$

- Congruence

$$\frac{\overline{s}_1 \longrightarrow \overline{s}'_1 \quad \cdots \quad \overline{s}_n \longrightarrow \overline{s}'_n}{\overline{f}(\overline{s}_1, \cdots \overline{s}_n) \longrightarrow \overline{f}(\overline{s}'_1, \cdots \overline{s}'_n)}\ f \in \Omega(\mathbf{n},\mathbf{1})$$

- Replacement

$$\frac{\overline{u}_1 \xrightarrow{T_1} \overline{v}_1 \quad \cdots \quad \overline{u}_n \xrightarrow{T_n} \overline{v}_n}{\overline{s}(\overline{u}_1, \cdots \overline{u}_n) \longrightarrow \overline{s}'(\overline{v}_1, \cdots \overline{v}_n)}\ A = (\overline{s}, \overline{s}')$$

3. MODEL OF A REWRITING THEORY

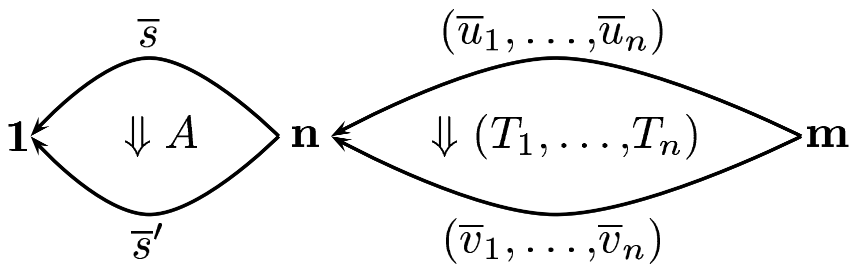
A rewrite theory $(\mathbb{S},R)$ is simulated on a 2-category $\mathcal{C}$, whose:

- the 0-cells are natural numbers $\mathbf{0}$, $\mathbf{1}$, $\ldots$ $\mathbf{n}\ldots$
- the 1-cells $\mathbf{1} \longleftarrow \mathbf{n}$ are terms in $\mathbb{S}(\mathbf{n},\mathbf{1})$,
- the 2-cells are rewriting modulo the replacement:

$$\begin{array}{ccc} \overline{s}(\overline{u}_1,\ldots,\overline{u}_n) & \xRightarrow{A(\overline{u}_1,\ldots,\overline{u}_n)} & \overline{s}'(\overline{u}_1,\ldots,\overline{u}_n) \\ {\scriptstyle \overline{s}(T_1,\ldots,T_n)}\Big\Downarrow & A(T_1,\ldots,T_n) = & \Big\Downarrow{\scriptstyle \overline{s}'(T_1,\ldots,T_n)} \\ \overline{s}(\overline{v}_1,\ldots,\overline{v}_n) & \xRightarrow[A(\overline{v}_1,\ldots,\overline{v}_n)]{} & \overline{s}'(\overline{v}_1,\ldots,\overline{v}_n) \end{array}$$

where $A = (\overline{s},\overline{s}') \in R$ and $\overline{u}_i \xrightarrow{T_i} \overline{v}_i$.

- the horizontal composition is given by:



$$A * (T_1,\ldots,T_n) := A(\overline{v}_1,\ldots,\overline{v}_n) + \overline{s}(T_1,\cdots,T_n) = \overline{s}'(T_1,\ldots,T_n) + A(\overline{u_1},\ldots,\overline{u}_n).$$

4. Crossed modules of algebraic theories

Definition. A crossed module of algebraic theories is an algebraic theory $\mathcal{C}$ (strongly) enriched in groupoid.

i.e. $\mathcal{C}$ consists of the following data:

- the 0-cells are the integers $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}, \ldots,$
- a groupoid $\mathcal{C}(\mathbf{n},\mathbf{m})$, for any 0-cells $\mathbf{n},\mathbf{m}$,
- an unit $1_{\mathbf{n}} \in \mathcal{C}(\mathbf{n},\mathbf{n})_0$, for any 0-cell $\mathbf{n}$,
- an associative and unitary functors $* : \mathcal{C}(\mathbf{n},\mathbf{p}) \times \mathcal{C}(\mathbf{p},\mathbf{m}) \longrightarrow \mathcal{C}(\mathbf{n},\mathbf{m})$, for any 0-cells $\mathbf{n},\mathbf{m},\mathbf{p}$

such that

i) 0-cells and 1-cells form an algebraic theory $\mathbb{S}$,

ii) each product diagram $\mathbf{n_1} \xleftarrow{\pi_1} \mathbf{n_1} \times \mathbf{n_2} \xrightarrow{\pi_2} \mathbf{n_2}$ in $\mathbb{S}$ induces an isomorphism of groupoids:

$$\mathcal{C}(\mathbf{m}, \mathbf{n}_1 \times \mathbf{n}_2) \simeq \mathcal{C}(\mathbf{m}, \mathbf{n}_1) \times \mathcal{C}(\mathbf{m}, \mathbf{n}_2)$$
$$w \longmapsto (\pi_1 w, \pi_2 w)$$
$$\alpha \longmapsto (\pi_1 \alpha, \pi_2 \alpha).$$

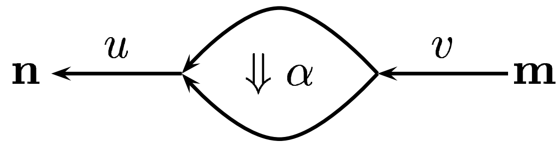
▶ **Remark** This notion was firstly considered by Mitchell ('72), and introduced by Baues ('90) (Track Categories) for small categories, and introduced by Baues-Jibladze for algebraic theories ('02) (Track Theories).

III. Identities among relations and Finite Derivation Type

1. The free crossed module: $\mathcal{R} = (\mathbb{F}(\Omega), R)$

The free crossed module $\mathcal{C} = C(R) \longrightarrow \mathbb{F}(\Omega)$ generated by a TRS $< \Omega \mid R >$ is defined by :

- 0-**cells** : integers **0**, **1**, $\cdots$, **n**, $\cdots$,
- 1-**cells** : terms $w \in \mathbb{F}(\Omega)$,
- 2-**cells** : Graph of whiskers $W(\mathbf{n},\mathbf{m})$: $\mathrm{Ob}(W(\mathbf{n},\mathbf{m})) = \mathbb{F}(\Omega)(\mathbf{n},\mathbf{m})$,

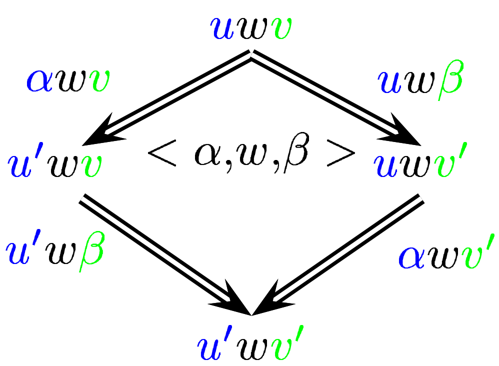
$\mathrm{Ar}(W(\mathbf{n},\mathbf{m})) \ni$ 

$$\alpha \in R \cup R^{-} \cup \{< \alpha_1, \ldots, \alpha_n > \mid \alpha_i \in R \cup R^{-}\} \; u,v \in \mathbb{F}(\Omega).$$

$$\mathcal{C}(\mathbf{n},\mathbf{m}) := \mathbf{F}(W(\mathbf{n},\mathbf{m}))/\mathrm{Rel}$$

where Rel is the set of relations :

- **inverse** : $u\alpha^{-1}v + u\alpha v = 0$, $u\alpha v + u\alpha^{-1}v = 0$,
- **product** : $\pi_i < \alpha_1, \ldots, \alpha_n >= \alpha_i$,
- **Peiffer** : $< \alpha, w, \beta >= 0$, where



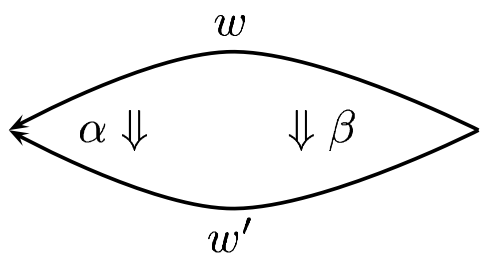
$u \xrightarrow{\alpha} u' \in R$

$v \xrightarrow{\beta} v' \in R$

$w \in \mathbb{F}(\Omega)$

▶ **Proposition.** A TRS $\mathcal{R}$ is orthogonal if an only if the crossed module $\mathcal{C}(\mathcal{R})$ is trivial.

i.e. any parallel 2-cells are equals.



▶ **Homotopy on crossed module**

Let $\mathrm{P}^{\|}(\mathcal{C})$ the set of parallel 2-cells in $\mathcal{C}$.

An homotopy on $\mathcal{C} = C(R) \rightrightarrows \mathbb{F}(\Omega)$ is an equivalence relation $\approx \subseteq \mathrm{P}^{\|}(\mathcal{C})$:

i) if $\alpha_1 \approx \alpha_2$, then $u\alpha_1 v \approx u\alpha_2 v$, for any $u, v \in \mathbb{F}(\Omega)$,

ii) if $\alpha_1 \approx \alpha_2$, then $\gamma + \alpha_1 + \beta \approx \gamma + \alpha_2 + \beta$, for any 2-cells γ, β.

An homotopy base on $\mathcal{C}$ is a subset B of $\mathrm{P}^{\|}(\mathcal{C})$ such that :

$$\approx_B = \mathrm{P}^{\|}(\mathcal{C}).$$

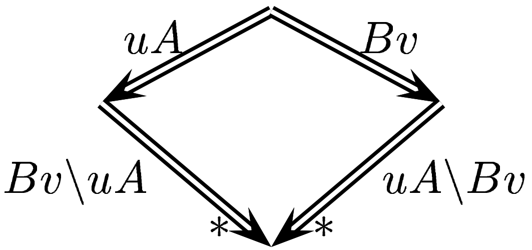
Definition. A TRS $< \Omega \mid R >$ has finite derivation type if the free crossed module $\mathcal{C} = C(R) \rightrightarrows \mathbb{F}(\Omega)$ has a finite homotopy base.

▶ Main Theorem

1. (Invariance) If $< \Omega \mid R >$ and $< \Omega' \mid R' >$ are Tietze equivalent, then $< \Omega \mid R >$ has finite derivation type if and only if $< \Omega' \mid R' >$ has finite derivation type.

2. If $< \Omega \mid R >$ is complete (noetherian and confluent), then the set

$\Big\{$



$\mid \quad \langle uA, Bv \rangle$ is a critical pair in $< \Omega \mid R > \Big\}$

is an homotopy base for $C(R) \rightrightarrows \mathbb{F}(\Omega)$

3. A finite complete term rewriting system has finite derivation type

2. Identities among relations

▶ To the free crossed module $\mathcal{C} = \; C(R) \rightrightarrows \mathbb{F}(\Omega)$, we associate a cartesian natural system of groups:

$$\begin{aligned} \mathrm{Aut}^{\mathcal{C}} : F(\mathbb{F}(\Omega)) &\longrightarrow \mathsf{Gp} \\ w &\longmapsto \mathrm{Aut}^{\mathcal{C}}_w = \{\text{endo-2-cell on } w\} \\ w \xrightarrow{(u,v)} w' &\longmapsto \mathrm{Aut}^{\mathcal{C}}_w \longrightarrow \mathrm{Aut}^{\mathcal{C}}_{w'} \\ &\qquad\quad \alpha \;\longmapsto\; u\alpha v \end{aligned}$$

▶ **Proposition.** The crossed module $\mathcal{C}$ is abelian, i.e. $Aut^{\mathcal{C}}_w$ is abelian for any w.

As consequence of [Baues-Jibladze'02] :

▶ **Corollary.** Let $\mathbb{A} = \mathbb{F}(\Omega)/R$ and $p : \mathbb{L}(\Omega) \longrightarrow \mathbb{A}$ be the canonical projection, there is a cartesian natural system $\Pi : F\mathbb{A} \longrightarrow \mathsf{Ab}$ defined by

$$\Pi_w := \mathbb{Z}[\, [\alpha] \mid \alpha \in \mathrm{Aut}_{\widetilde{w}},\; p(\widetilde{w}) = w \,]/\{[\alpha] + [\beta] = [\alpha + \beta], [\alpha + \beta] = [\beta + \alpha]\}$$

and an isomorphim of cartesian natural systems :

$$\varphi_- : \Pi_p \xrightarrow{\sim} \mathrm{Aut}^{\mathcal{C}} \quad \text{in} \quad \mathsf{Ab}^{F(\mathbb{F}(\Omega))}$$
$$\varphi_w(\alpha) = [\alpha], \quad \text{for any } w \in \mathbb{L}(\Omega).$$

Definition. Π is called the cartesian natural system of identities among the relations for $< \Omega \mid R >$.

▶ **Proposition.** The abelianisation map

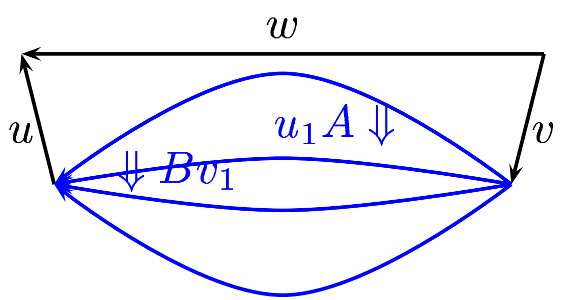
$$(-)_{ab} : C(R) \longrightarrow \mathbb{Z}\mathbb{A}[R]$$
$$u\alpha v \longmapsto (\overline{u},[\alpha],\overline{v})$$

induces an isomorphism of cartesian natural system:

$$\varphi : \Pi \xrightarrow{\sim} \mathrm{H}$$

where H is the cartesian natural system of 2-homological syzygies generated by confluence diagram induced by critical pairs :

H_w is generated by factorisation of w by head of critical pairs



$$w_1 v_1 \xrightarrow{A} v_2, \quad u_1 w_1 \xrightarrow{B} u_2 \in R, \qquad w \in \mathbb{A}.$$

▶ **Main Theorem on the equivalence of Squier's finiteness conditions**

The homological and homotopical finiteness conditions are equivalents.

Corollary. Let $\mathbb{A}$ be an algebraic theory. The following assertions are equivalent :

i) $\mathbb{A}$ has finite derivation type,

ii) $\mathbb{A}$ is of type FP_3 in $\mathsf{Ab}^{F\underline{\mathbb{A}}}$.

▶ The next steps consist in the construction of a free resolution of Π,

by cartesian naturals systems involving higher order overlappings.

3. Problems

► **Problem 1.** Consider free crossed modules for rewriting modulo order-sorted algebraic theories

- $\lambda\sigma$-calculus : Terms, Substitution

- ChAM : Solution, Molecule, Molecules

Introduce an appropriated notion of abelian coefficients.

► **Problem 2.** Implement the algorithm for computing identities among relations and higher syzygies induced by the free crossed modules of algebraic theories.