A Homotopic Approach to Mesh Generation

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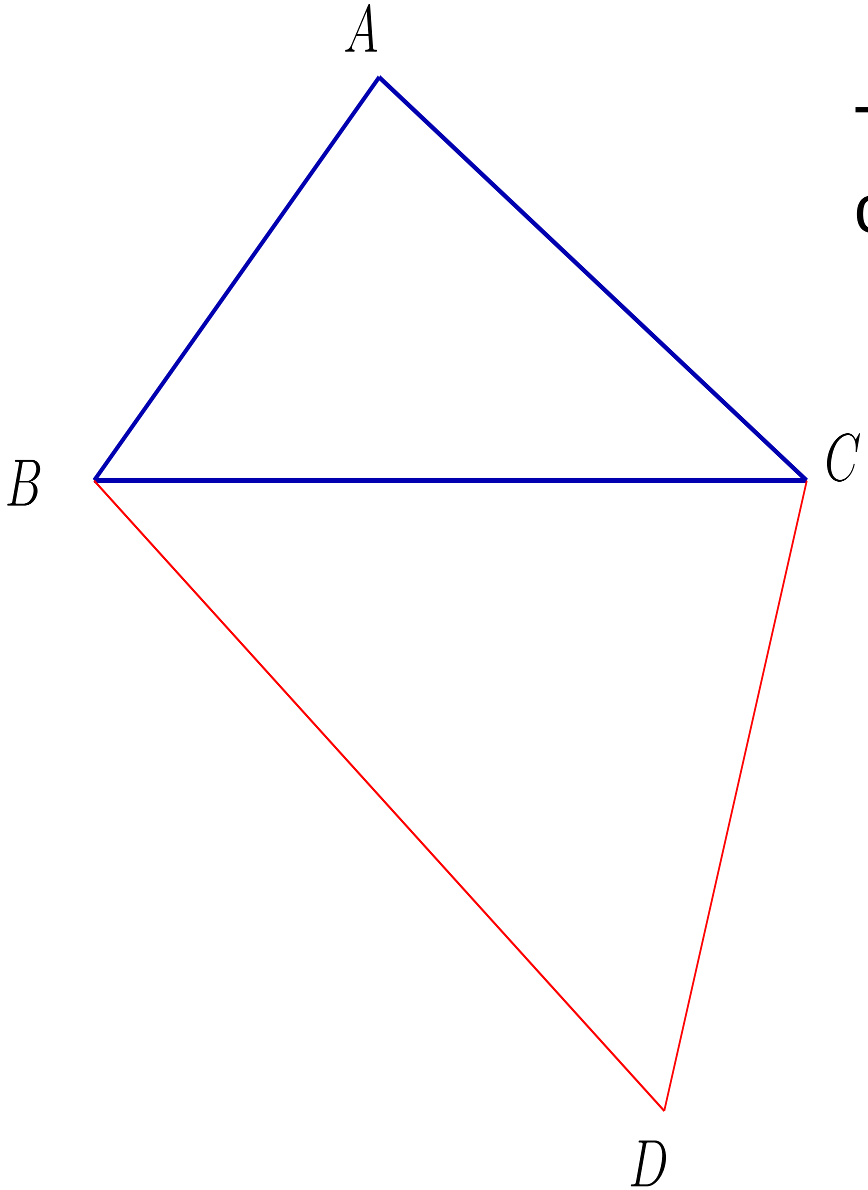
Delaunay Triangulations

A triangulation of a region Ω (possibly non-convex and/or non-simply connected) in $\mathbb{R}^2$ is is called Delaunay if the circumcircle of any triangle does not contain any other vertices in its interior.

- The dual of a Voronoi Diagram is a Delaunay Triangulation.
- Given a set of points in Ω, there is a unique Delaunay Triangulation having the points as vertices.
- Among all triangulations having the same set of vertices, the Delaunay Triangulation is the one with the maximum minimum angle.
- A triangulation of a convex region Ω is Delaunay if and only if it locally Delaunay.



Then D is not in the circumdisc of $\triangle ABC$ iff:

$$\begin{vmatrix} 1 & a_1 & a_2 & a_1^2 + a_2^2 \\ 1 & b_1 & b_2 & b_1^2 + b_2^2 \\ 1 & c_1 & c_2 & c_1^2 + c_2^2 \\ 1 & d_1 & d_2 & d_1^2 + d_2^2 \end{vmatrix} > 0$$

Alternatively,

$$\boxed{\angle BDC < \pi - \angle BAC}.$$

Parallel Mesh Generation

Requirements for a parallel mesh generation algorithm:

- **Efficiency and Scalability**:
  — Small communication, synchronization, and load imbalancing costs;
- **Stability**:
  — distributed meshes should retain the **same quality of the elements** and partition properties as sequentially generated and partitioned meshes;
- **Simple Domain decomposition**:
  — Inexpensive and no new ”smaller” artificial features;
- **Re-use Scalar Codes**:
  — It takes 5-10 years to develop sequential algorithms and libraries. Improvements are open-ended.

Overview

- **A priori** meshing of subproblem interfaces (Lohner89, Galtier97, Said97)
  Mesh the interfaces of the subproblems and then mesh, in parallel, the individual subproblems.
- **A posteriori** meshing interfaces (Shephard99, Lohner99)
  First solve the meshing problem in each of the subproblems in parallel, and then mesh the interfaces so that the global mesh is consistent.
- Simultaneous meshing and partitioning of subproblems and interfaces (Chrisochoides97 and ’99)
  — Objective: stability

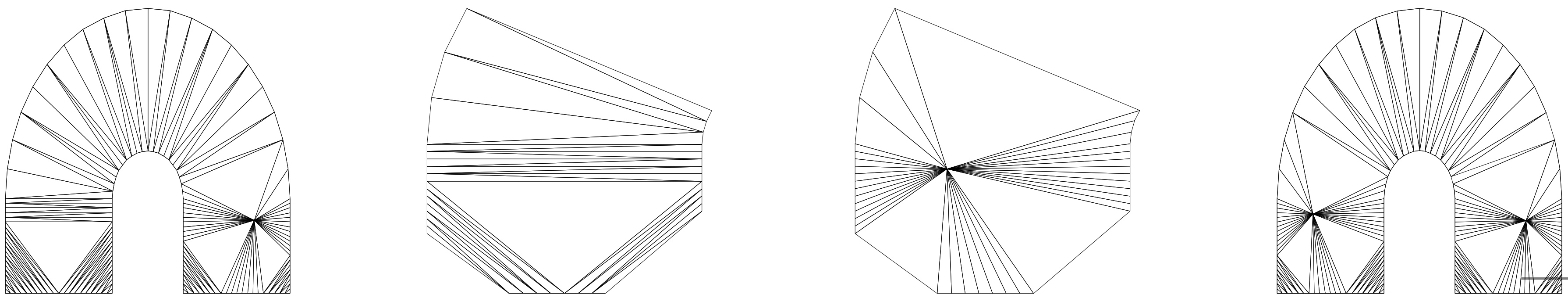
Algorithms

For parallel mesh generation, the most appropriate algorithms are incremental algorithms. These are algorithms that the refinement is done by adding points in the original triangulation (usually one at the time) and re-triangulating the result. The algorithm of choice is the Bowyer–Watson Algorithm:

Bowyer-Watson Delaunay Kernel

- point creation: a new point is created using an appropriate spatial distribution technique
- cavity computation: triangles that violate the *empty sphere criterion* are removed
- element creation: new triangles are built by properly connecting the newly created point with boundary points of the cavity.

M_i Mesh | Point Insertion & comput. of C_i | Triang. of C_i | New Mesh M_{i+1}

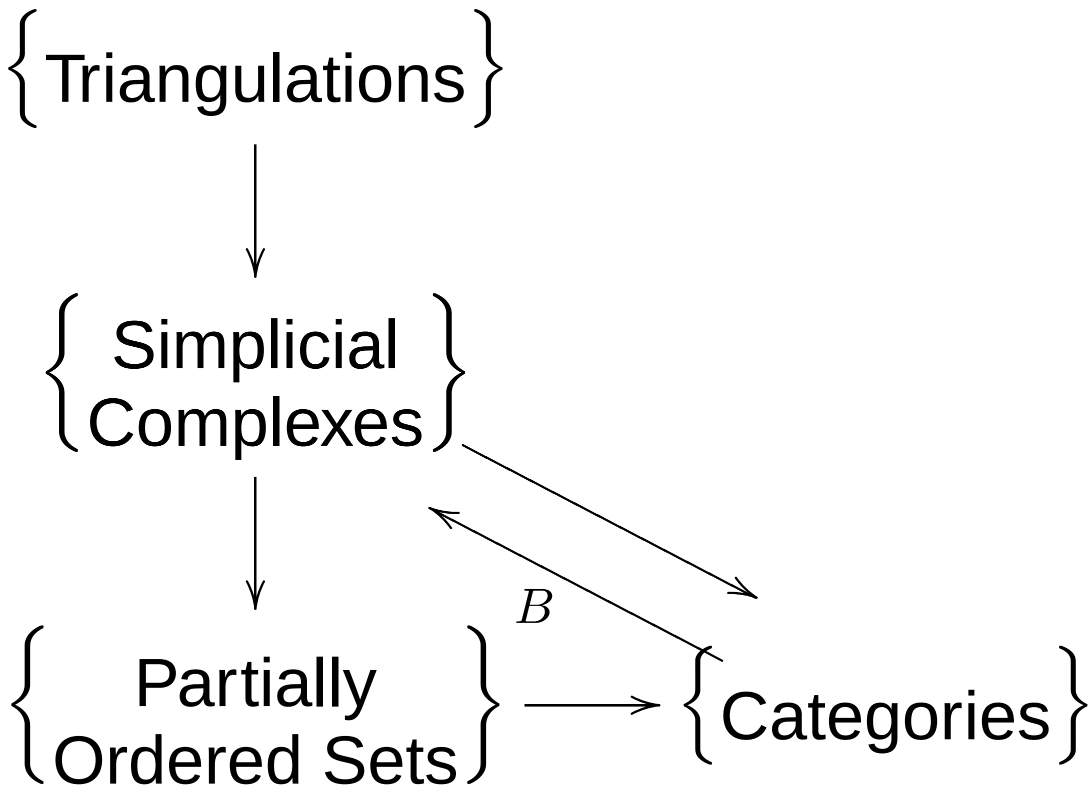


Parallel Guaranteed Quality Meshing

- Is it possible to prove termination and correctness in the context of concurrency (i.e., partial order)?
- Need mathematical framework to argue about quality and concurrency at the same time.

Categorization of Meshes

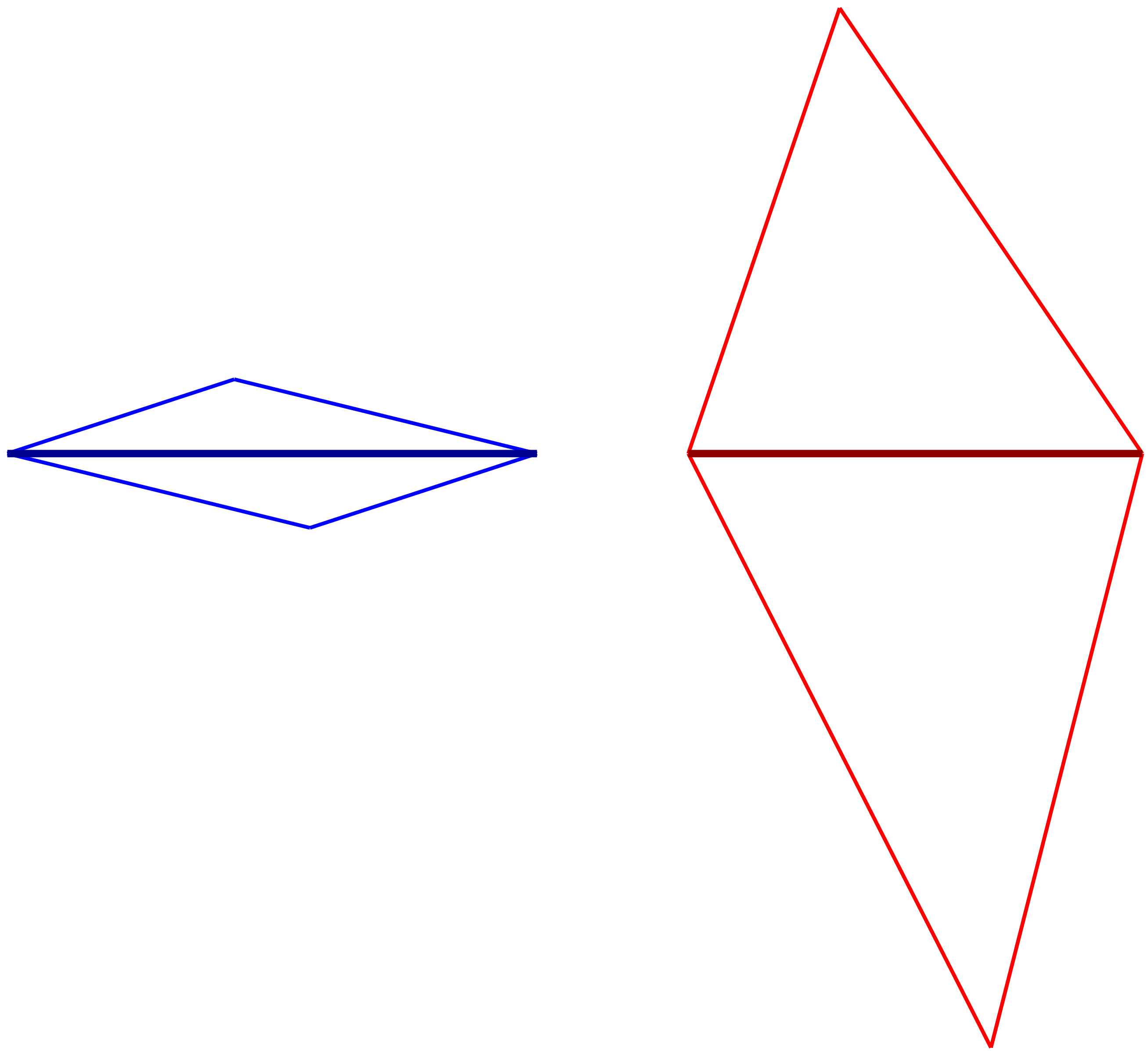
There is a naive correspondence



The partial order structure in a triangulation is given by the obvious:

$$\text{vertex} \leq \text{edge} \leq \text{face}$$

The problem is that such a simplistic approach does not capture the geometric properties of the triangulation. It captures only its topological properties.



Let $\mathcal{I}so\mathcal{R}ep$ be the category of pairs (G, ρ) where

- G is a group,
- $\rho : G \to \mathrm{Iso}(\mathbb{R}^2)$ a faithful representation of G into the group of the Euclidean Isometries of $\mathbb{R}^2$.

We write $\mathcal{FPI}so\mathcal{R}ep$ the category of finite products of objects of $\mathcal{I}so\mathcal{R}ep$.

Let $\mathcal{P}$ be the partially ordered set of the elements of a triangulations, considered as a category.

A mesh generation is a functor:

$$\mathbb{F} : \mathcal{P}^{op} \longrightarrow \mathcal{FPI}so\mathcal{R}ep$$

such that:

- For each $v \in V$ of degree d_v, $\mathbb{F}(v) = (G, \rho)$, where

$$G = \prod_{i=1}^{d_v - 1} G_i, \quad \rho = \prod_{i=1}^{d_v - 1} \rho_i$$

  and each G_i is a dihedral group and Im(ρ_i) is the dihedral group generated by two reflections. All the axes of the reflections intersect at the same point.
- For each $e \in E$, $\mathbb{F}(e) = (\mathbb{Z}/2\mathbb{Z}, \rho)$, where ρ maps the non-trivial element to a reflection about an axis ℓ_x.
- The image of a face under $\mathbb{F}$ is the trivial group.

On the morphisms:

If v is an vertex of an edge e, then $\mathbb{F}(e \geq v)$ is induced by the inclusion map.

With the extra structure given on $\mathcal{P}$, the functor $\mathbb{F}$ induces a map:

$$\phi : |\mathcal{P}^{op}| = \text{hocolim}_{\mathcal{P}^{op}} * \longrightarrow \mathbb{R}^2$$

that maps vertices to points. The image of the map is a triangulation.

Conversely, given a pair $(\mathcal{P}, \mathbb{F})$ so that $\mathcal{P}$ is a partially ordered set such that

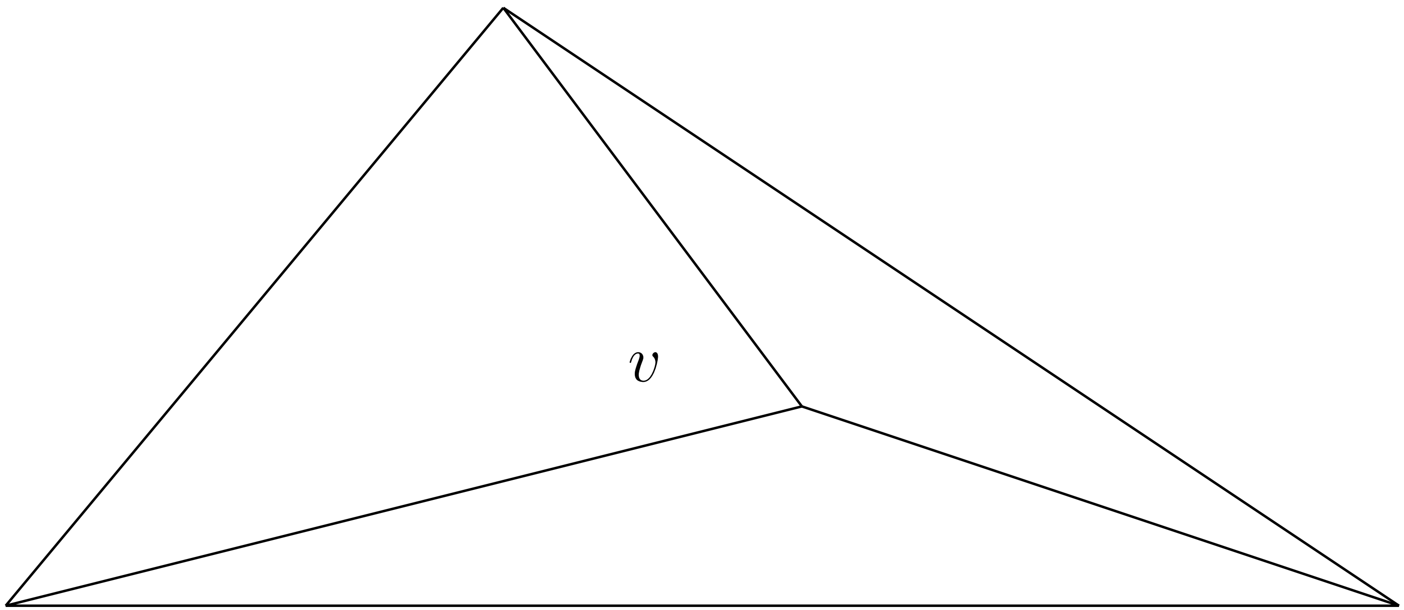
- The geometric realization $|\mathcal{P}|$ is a triangulation of a planar region.
- The triangles determined by $\mathbb{F}$ are Euclidean.

Categorical Description of the B–W

Let $(\mathcal{P}_0, \mathbb{F}_0)$ be a Delaunay Mesh.
(1) Insert a point. Change

$$\boxed{V(\mathcal{P}) \longrightarrow V(\mathcal{P}) \cup \{v\}}$$

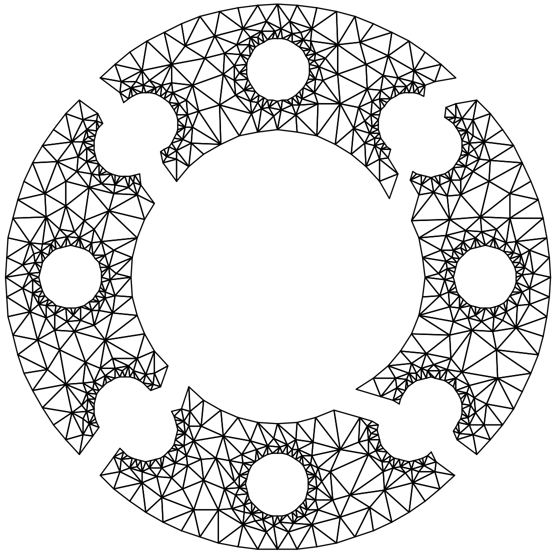
That is determined by a triangle and two dihedral groups

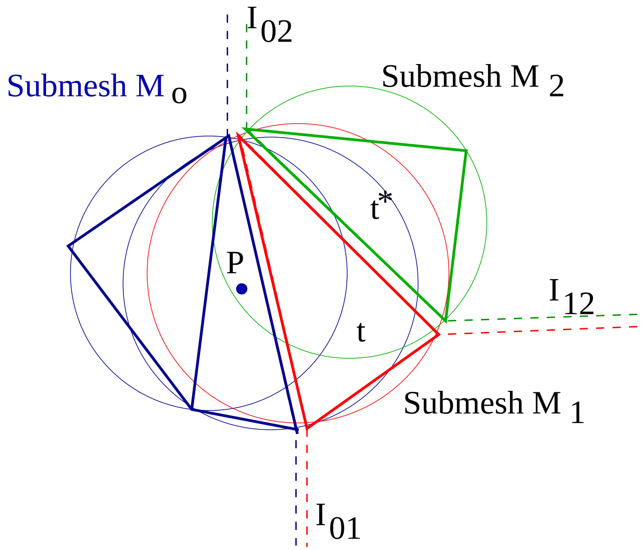


(2) Determine the cavity C_v. That is done using the angle criterion to determine the triangles of the cavity.
(3) "Remove" the cavity. The result is a partially ordered set $\mathcal{P}'$ and a functor $\mathbb{F}'$ defined on all the vertices and edges, with $\mathbb{F}'(v)$ the trivial group.
(4) "Re-triangulate" the ball of v. That is done by adding the edges in the cavity of v i.e., determining the corresponding dihedral angles. That determines the next step $(\mathcal{P}_1, \mathbb{F}_1)$.

All the calculations involve solving systems of linear equations.

SMGP




I_{02}
Submesh M_o
Submesh M_2
t^*
P
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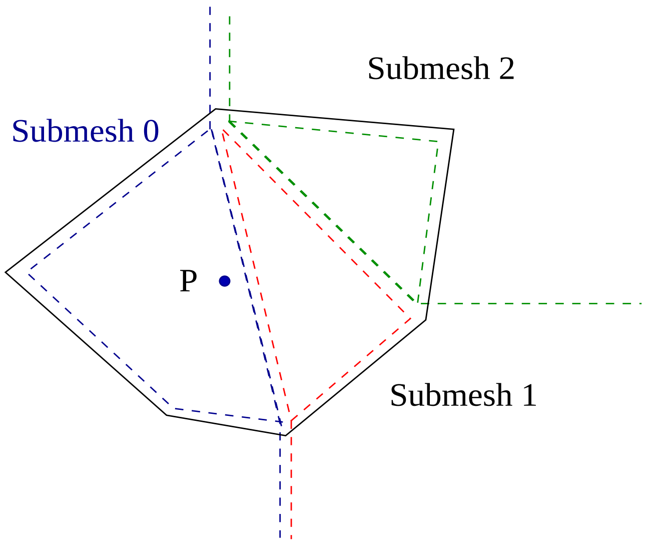
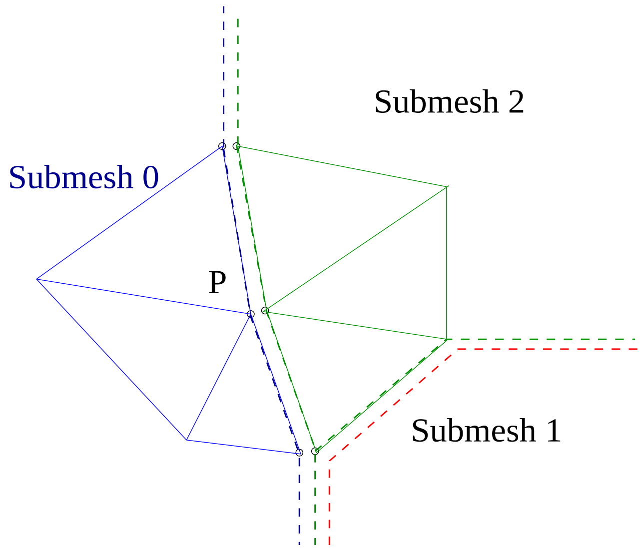
Submesh M_1
I_{01}


Submesh 2
Submesh 0
P
Submesh 1


Submesh 2
Submesh 0
P
Submesh 1

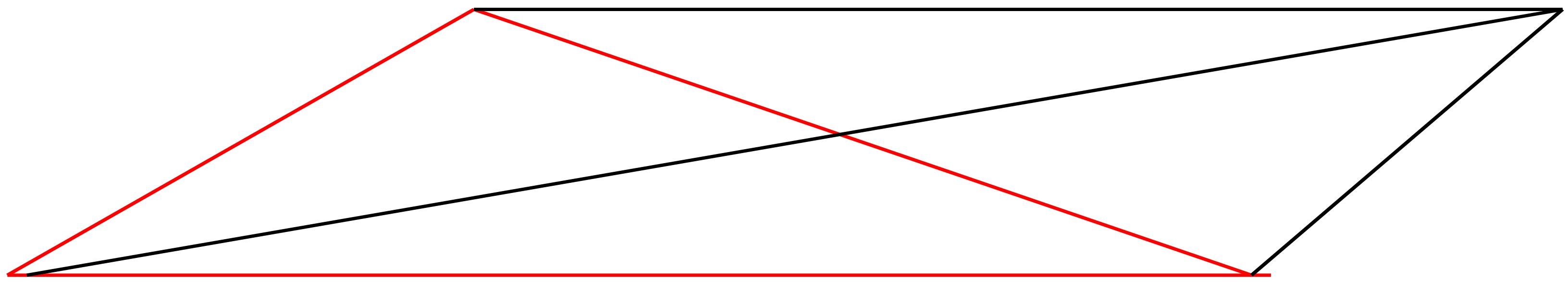
Extension to Dim. 3

In this case, $(\mathcal{P}, \mathbb{F})$ is a pair such that:

- The geometric realization $|\mathcal{P}|$ is a tetrahedration of a body in $\mathbb{R}^3$.
- The functor $\mathbb{F}$ assigns:

vertices	$\longrightarrow$	products of dihedral groups,
edges	$\longrightarrow$	dihedral group,
faces	$\longrightarrow$	$\mathbb{Z}/2\mathbb{Z}$,
tetrahedra	$\longrightarrow$	trivial group.

Slivers

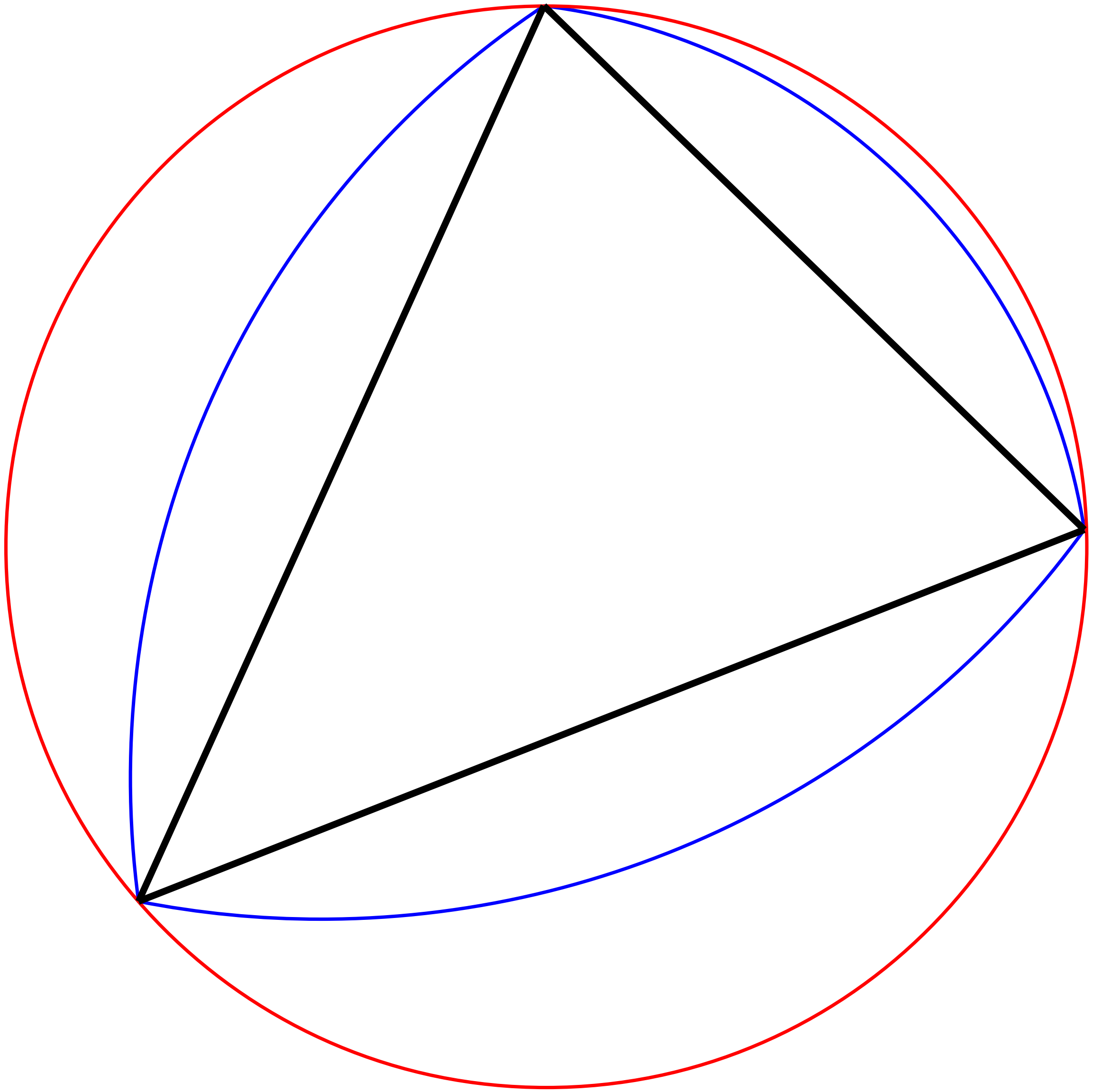


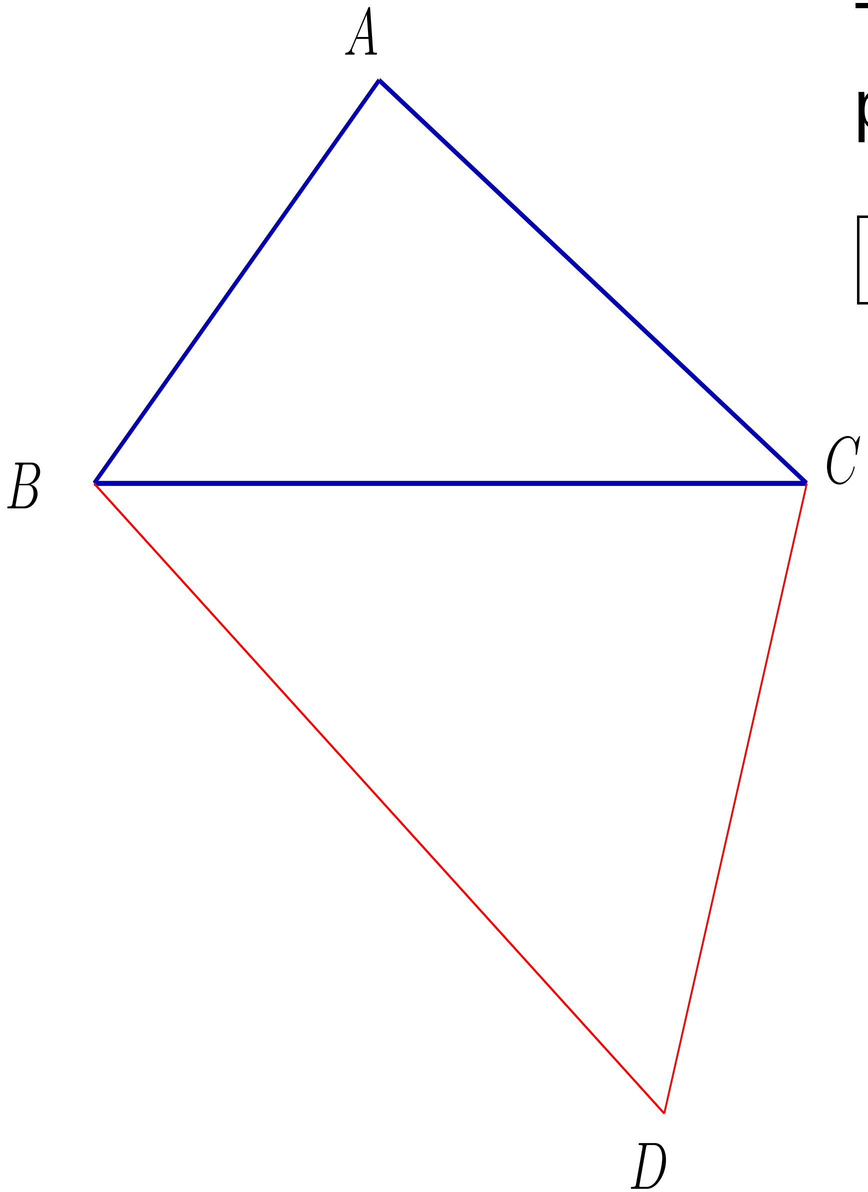
- Distance from a vertex to the opposite plane is small.
- The radius-edge ratio and the volume-cube of edge ratio are both small. The aspect ratio (circumradius over inradius) is large.
- One dihedral angle is very large.

Floating Point Issues

Fortune suggested a modification to deal with floating point arithmetic. He suggested the notion of α-pseudosphere of a triangle for some $\alpha > 0$ and he relaxed the Delaunay condition to:

The α-pseudosphere of every triangle is empty.





Then D is not in the α-pseudodisc of $\triangle ABC$ iff:

$$\boxed{\angle BDC < \pi - \angle BAC + \alpha}.$$

Single Space Description

Alternatively, we can "describe" the triangulation as a functor:

$$\mathbb{F} : \mathcal{P}^{op} \to \mathcal{FPS}ub(O(2))$$

to the category of finite products of subgroups of the group of linear isometries of $\mathbb{R}^2$.

Composing with a $classifying$ space functor:

$$\mathcal{P}^{op} \xrightarrow{\mathbb{F}} \mathcal{FPS}ub(O(2)) \hookrightarrow \mathcal{G}roups \xrightarrow{\mathcal{B}} \mathcal{S}paces$$

where $\mathcal{B}(G)$ is the metric space $\mathbb{R}^2/G$ with the induced metric.

The mesh space is defined to be the metric space:

$$\mathcal{T}(\mathcal{P}, \mathbb{F}) = \text{hocolim}_{\mathcal{P}^{op}}(\mathcal{B} \circ \mathbb{F}).$$

For a functor:

$$\Phi : \mathcal{P}^{op} \longrightarrow \mathcal{S}paces,$$

the homotopy colimit is defined as:

$$\text{hocolim}_{\mathcal{P}^{op}} \Phi = \coprod_{\sigma \in |\mathcal{P}^{op}|} \sigma \times \Phi(\sigma) / \sim, \quad (t, x) \sim (t, \Phi(x)),$$

for $t \in \tau$, $x \in \Phi(\sigma)$, and $\sigma \geq \tau$,

Similarity Structures

Rivin used angle assignments to study the Realization Problem:

When is a angle assignment to a triangulation of a singular surface can be realized by a similarity structure on the surface?

He showed that if an angle assignment satisfies certain obvious necessary conditions and it is Delaunay then it is realizable.

Furthermore, similar methods were used in calculations of moduli spaces of such structures.

In this case, the functor $\mathbb{F}$ takes values in the category of similarities of the Euclidean Space.

Hyperbolic Triangulations

Thurston used disc patterns to study uniform structures on surfaces (Andreev–Thurston Theorem).
Leibon (extending Thurston's work) studied the discrete analogue and:

- Characterized the uniform structure of hyperbolic surfaces using Delaunay Triangulations.
- Proved the Discrete Andreev–Thurston Theorem.
- Proved the Gauss–Bonnet Theorem.
- Derived the continuous theorems as limits of the discrete ones.

In this case, the functor $\mathbb{F}$ takes values in the category of hyperbolic isometries.